Operations Research Methods for Multi-Domain Campaign Phase Planning

THESIS

John Andrew Schlicht, CPT, USA

AFIT-ENS-MS-19-M-148

DEPARTMENT OF THE AIR FORCE
AIR UNIVERSITY

AIR FORCE INSTITUTE OF TECHNOLOGY

Wright-Patterson Air Force Base, Ohio

AFIT-ENS-MS-19-M-148

OPERATIONS RESEARCH METHODS FOR MULTI-DOMAIN CAMPAIGN PHASE PLANNING

THESIS

Presented to the Faculty

Department of Operational Sciences

Graduate School of Engineering and Management

Air Force Institute of Technology

Air University

Air Education and Training Command

in Partial Fulfillment of the Requirements for the

Degree of Master of Science in Operations Research

John Andrew Schlicht, B.S.M., M.S.E.M.

CPT, USA

March 21, 2019

AFIT-ENS-MS-19-M-148

OPERATIONS RESEARCH METHODS FOR
MULTI-DOMAIN CAMPAIGN PHASE PLANNING

THESIS

John Andrew Schlicht, B.S.M., M.S.E.M.
CPT, USA

Committee Membership:

Dr. Richard F. Deckro
Chair

Lt Col Matthew J. Robbins, Ph.D.
Member

AFIT-ENS-MS-19-M-148

Abstract

In “Antiaccess Warfare as Strategy,” Tangredi posits the question and need to consider multiple domains and governmental and warfighting functions in various phases of campaign execution. Multi-domain integration within and across various phases of the joint campaign presents a host of non-linear factors that are compounded and amplified by uncertainties. Colonel Blotto is a simple game that is suited to compare traditional force-on-force military engagements where mass wins the day, but have had limited application to more complex military planning. This thesis explores the formulation schema, data-driven parameters, methods of calculation, and scenarios applicable to a generalized Colonel Blotto (General Blotto) game. It explores this generalized game theory framework, its applicability to multi-domain operations, and recommends future research areas that could help to extend its applicability, enabling planners and commanders to gain similar insight as to those straight-forward applications.

Acknowledgements

To my loving and supportive spouse, thank you for all of your sacrifices.

To my advisor, Dr. Deckro, thank you for your help, guidance, counsel, support, and patience.

To Lt Col Robbins, thank you for your critical feedback, questions, and attention to detail.

John Andrew Schlicht

Table of Contents

Page

Page

List of Figures

Figure		Page

List of Tables

List of Notation

$S = S_1 \times S_2$: the set of all possible strategy profiles.

$S_1 \equiv \{s_1^1, s_1^2, \ldots, s_1^j\}$: the strategy set (space) of Player 1.

$S_2 \equiv \{s_2^1, s_2^2, \ldots, s_2^k\}$: the strategy set (space) of Player 2.

X : Player 1 in the Colonel Blotto game.

Y : Player 2 in the Colonel Blotto game.

$\sigma = (\sigma_1, \sigma_2)$: the mixed strategy profile of the two players.

$\sigma_1 = (p_1^1, p_1^2, \ldots, p_1^j)$: a mixed strategy available to Player 1; a probability distribution over the j pure strategies in S_1.

$\sigma_2 = (p_2^1, p_2^2, \ldots, p_2^k)$: a mixed strategy available to Player 2 a probability distribution over the k pure strategies in S_2.

θ_i : $\theta_i \in \mathbb{R}, 0 \leq \theta_i \leq 1, \sum \theta_i = 1$; the bayesian probability that player i plays a pure strategy profile, s_i.

$j = \frac{(n_1+m-1)!}{(m-1)!n_1!}$: the number of strategies available to Player 1; the number of ways to partition n_1 resources on m fronts.

$k = \frac{(n_2+m-1)!}{(m-1)!n_2!}$: the number of strategies available to Player 2; the number of ways to partition n_2 resources on m fronts.

m : $m \in \mathbb{N}, m \geq 2$; the number of fronts.

n : $n \in \mathbb{N}, n \geq 2$; the number of resources available to be allocated between fronts of a symmetric Blotto game.

n_i : the number of resources available to be allocated by player i between fronts of an asymmetric Blotto game.

p_1 : $p_1 \in \mathbb{R}, 0 \leq p_1 \leq 1, \sum_j p_1^j = 1$; the frequentist probability that Player 1 plays a pure strategy profile, s_1.

p_2 : $p_2 \in \mathbb{R}, 0 \leq p_2 \leq 1, \sum_k p_2^k = 1$; the frequentist probability that Player 2 plays a pure strategy profile, s_2.

$s = (s_1, s_2)$: $s \in S$: the strategy profile (a vector of strategies) of the two players; the "strategic state" of a game.

$s_1 = (x_1, x_2, \ldots, x_m)$: a pure strategy profile (vector of resource allocations to each of m fronts) available to Player 1.

$s_2 = (y_1, y_2, \ldots, y_m)$: a pure strategy profile (vector of resource allocations to each of m fronts) available to Player 2.

u_i $|S \rightarrow \mathbb{R}$; the utility realized for player i where $u_i(s)$ is the payoff function for player i.

w_i^m : weight applied to each front as an ordinal utility of winning the front realized by player i.

x_i^{CB} : Player 1's budget allocation to front i in the Colonel Blotto game.

x_i : $x_i \in \mathbb{N}_{\geq 0}, \sum_m x_m = n_1$; the integer number of resources Player 1 budgets to front i, from 0 to n_1 in the General Blotto game.

y_i^{CB} : Player 2's budget allocation to front i in the Colonel Blotto game.

y_i : $y_i \in \mathbb{N}_{\geq 0}, \sum_m y_m = n_2$; the integer number of resources Player 2 budgets to front i, from 0 to n_2 in the General Blotto game.

List of Acronyms

A2/AD Antiaccess / Area Denial

COA course of action

COG Center of Gravity

DIME Diplomatic, Information, Military, and Economic

DSS Decision Support System

FAA-DC Feasible, Acceptable, Adequate, Distinguishable, and Complete

JFC Joint Force Commander

JIPOE Joint Intelligence Preparation of the Operational Environment

JP Joint Publication

JPP Joint Planning Process

OPLAN operation plan

PLA People's Liberation Army (military forces of the Chinese Communist Party)

PMESII Political, Military, Economic, Social, Infrastructure, and Information

TTP Tactics, Techniques, and Procedures

VFT Value Focused Thinking

OPERATIONS RESEARCH METHODS FOR
MULTI-DOMAIN CAMPAIGN PHASE PLANNING

I. Introduction

> *I tell this story to illustrate the truth of the statement I heard long ago in the Army: Plans are worthless, but planning is everything. There is a very great distinction because when you are planning for an emergency you must start with this one thing: the very definition of "emergency" is that it is unexpected, therefore it is not going to happen the way you are planning.*
>
> - Dwight D. Eisenhower[1]

1.1 Background

As the lone global superpower with currently little direct risk to its homeland from a near-peer or regional power, the United States is in a unique position of both power and responsibility in the world. The ability to reliably project power across the globe has allowed the United States to pursue its economic, political, and moral interests as it has seen fit since the end of World War II. Some countries have come to rely upon the stability that this force projection provides, while others have sought to limit it in the hopes of pursuing their own regional ambitions. Antiaccess strategies

[1]During a speech in November 1957, Eisenhower told an anecdote about the maps used during U.S. military training. Maps of the Alsace-Lorraine area of Europe were used during instruction before World War I, but educational reformers decided that the location was not relevant to American forces. The maps were switched to a new location within the U.S. for planning exercises. A few years later, the military was deployed and fighting in the Alsace-Lorraine. Plans created in training may not have been correct, but the planning process on the same terrain proved crucial to the selection of appropriate actions as reality unfolded. Quoted from: Eisenhower, Dwight D. "Containing the Public Messages, Speeches, and Statements of the President, Remarks at the National Defense Executive Reserve Conference." Washington DC: Federal Register Division, National Archives and Records Service, General Services Administration. November 14, 1957:818.

seek to inhibit the power projecting capabilities of a global power by denying access, degrading lines of communication, or both. This thesis offers a method for planners and commanders to visualize various strategy sets to develop courses of action (COAs) to counter a potential adversary's emerging antiaccess strategies.

Modern operations research came of age in World War II, the result of strategic naval and aviation research in World War I and mathematical insights made since the turn of the century [28]. There is no doubt that scientific analysis has changed the conduct and strategy of warfare. Operations research methods are often used to tackle specific, narrowly defined problems such as "what is the best color to paint a plane" or "what is the most efficient way to get supplies from point A to point B?" Less often are such methods used to study the processes themselves. George Box famously said, "all models are wrong but some are useful [6]," further explaining that the primary question concerning the model is if it is "illuminating and useful?" Box refers to the idea that one does not necessarily have to account for every possible variable that could be identified in real life in order to create a mathematical model that is useful in approximating or explaining an interaction or phenomenon or to gain an understanding of the processes or probable future attributes of a system. In the strategic sense, theater campaign planning is concerned with developing insights into understanding system behavior which can help to guide effective planning and decision-making.

As a general matter, warfare has no rules. Granted, treaties, conventions, and norms theoretically provide frameworks for what is and is not acceptable in the conduct of warfare between nation states, but warfare is really nothing more than sanctioned killing and destruction through extreme violence. Tempo, audacity, mass, will, risk, and deceit are all important aspects of warfare that must be effectively leveraged to quickly and decisively bring about a victorious end to hostilities. Or are they? When decisive victory cannot be achieved, de facto victory through other means may be feasible. For example, asymmetric warfare embraces the "long game" at the cost of time, lives, productivity, and security. It is the ultimate battle of wills because it tests the will of a stronger power to stay invested (literally, and often at great expense) for what may be perpetuity. Because there are no rules in warfare, it is difficult to set up as a game. Games can be accurately simulated (chess, go, Risk, etc.) because they have defined rules.

Any discussion on strategy and warfare would be remiss without mentioning Carl von Clausewitz. To Clausewitz [9, vol. I bk. II ch. III], war did not belong entirely in the realm of art nor entirely in the realm of science. He argued that the goal of science is to discover knowledge and certainty, while art is focused on expressing creativity. As art involves some science and productive science involves creativity, they are not opposing ideas. Thus, an effective execution of war must involve the application of both art and science. He explained "ideal war" [9, vol. I bk. I ch. I] as the logical abstraction of war (hence ideal). This would be seen as a version that would be more easily modeled. He explained "real war" [9, vol. I bk. I ch. II] as the messy, unpredictable war that is subject to wills, personalities (competence and luck), third party involvement, the weather, political realities, and seeming acts of the gods. Clausewitz's trinity [9, vol. I bk. I ch. I sec. 28] emphasizes three connective tendencies: passion in the people and soldiers, chance in relation to the generals and strategy, and reason in the governmental and political processes and institutions. The trinity holds the disrupting factors that Clausewitz considered integral to the nature of war: uncertainty, danger, fear, courage, chance, and friction. Although these will not be measured directly, they will be considered by incorporating probability.

If warfare is messy, has few rules, is hard to model, and is ultimately the art of battling wills, personalities, and chance, but a scientific approach like operations research has yielded benefits, then where does that leave us? Tactical units exercise doctrine and Tactics, Techniques, and Proceduress (TTPs) at major exercises such as RIMPAC (Navy) or Red Flag (Air Force), rotations through the National Training Center (Army), or even by utilizing tabletop or computer simulation wargames. Although these are valuable at all echelons, they are also major investments in time, money, and human and materiel resources. Because of the investments required, they are also difficult to run iteratively and quickly learn lessons. Just as physically training tactical units helps to make them agile and adaptable, understanding potential strategies and being able to identify triggers that may point to the use of those explored strategies can help to train and prepare planners and commanders to be agile and adaptive - to use science to enable and accelerate art. This thesis proposes planners incorporate game theory into the Joint Planning Process (JPP) to study probable enemy actions and counter-actions to aide in identifying decision points and contemplate branch plans. Strategic agility is hard, but it is much easier if planned for.

Borel [5] introduced a two-person zero-sum game in which the players simultaneously distribute limited resources over several fronts with the goal of winning a majority of the fronts by having more resources than the opponent. He categorized it as an example of a game in which "the psychology of the players matters [5, 99]." The game was revisited after World War II with the advent of Operations Research and coined the "Colonel Blotto game" after being described by Gross and Wagner [16] as a game in which the fictitious "Colonel Blotto" was tasked with finding the optimum distribution of resources over n fronts knowing that:

1. on each front the player that has allocated the most resources will win, but
2. neither player knows how many resources the opposing player will allocate to each front, and
3. both players seek to maximize the number of fronts they expect to win.

By relaxing the rules, a generalized form of the game, General Blotto, has had some resurgence in popularity and study. Properly defined and incorporating probability, reason, mass, time, and space as needed and as appropriate, General Blotto may have strategic value in balancing creative art and measured science in military planning. After all, "The purpose of mathematical programming is insight, not numbers [12]."

1.2 Problem Statement

In "Antiaccess Warfare as Strategy," [40] Tangredi posits the question and need to consider multiple domains and governmental and warfighting functions in various phases of campaign execution. Multi-domain integration within and across various phases of the joint campaign presents a host of non-linear factors that are compounded and amplified by uncertainties. The need to have a plan and account for multiple domains in various phases of campaign execution is essential [23]. This thesis devises and demonstrates a proof-of-concept that offers a method to identify, expose, and resolve major sources of uncertainty in susceptible pathways and time horizons in decision support scenarios for warfighting staffs in meeting their future implementation and execution planning objectives. It explores the formulation schema, data-driven parameters, methods of calculation, and scenarios applicable to the design and construction of a suitable proof of concept to apply a generalized version of the classic Colonel Blotto (General Blotto) game to future campaign phase planning and time-sensitive operational environments.

1.3 Problem Approach

The task of planning a theater campaign across the warfighting domains and multiple phases is a daunting one. Assessing a situation through detailed modeling and simulation is an important aspect of military planning. Quickly assessing general strategic implications of a simplified model is also important for planners and senior leaders to begin to tackle a complex problem. This thesis utilizes a generalized version of the classic Colonel Blotto game to understand adversarial equilibriums and identify areas of opportunity or vulnerability with the intent that further detailed assessment is required. The intent is not to replace full-scale, detailed models, but for a simplified auxiliary model to supplement them [12].

1.4 Research Scope

This thesis utilizes a game theoretic approach to analyze campaign plans for situations with complex, uncertain, non-linear, endogenous and exogenous factors in multiple domains and across multiple phases. It is motivated by Tangredi's [40] argument for the need to consider multiple domains and all available resources in the various phases of campaign execution. For example, the need to balance the strategic allocation of resources in one (or some) theater(s) against the risk of being outflanked in another theater. The scope of this thesis is to design and construct a suitable proof of concept through schema formulation, data-driven parameters, methods of calculation, scenarios, and other considerations.

1.5 Assumptions

To be useful, a model should accurately represent a system only at the level of detail that is required to elicit the objective insight. To create an agile, adaptable, and useful model, reasonable assumptions must be made. This thesis makes the following assumptions to provide examples of models that are useful to planners and commanders because they provide broad insights, not detailed or specific answers.

As wills are difficult to measure, they are equally difficult to model. Will and morale are assumed to be outside the scope of this initial research effort, and only capabilities are modeled and compared to judge likely outcomes. Though warfare is often a battle of wills, it is assumed that it is a battle of physically measurable capabilities.

It is assumed capabilities can be measured or assessed. Although an adversary worthy of consideration is also assumed to be engaged in deception, for the purposes of planning, only known-knowns or known-unknowns are incorporated. In most situations that are carefully analyzed, it is the unknown-unknowns that cause significant deviations, introduce extreme variance, or come across as "random" events. These unknown-unknowns may be unknown to one or both sides and include weather events, third party interference, deception operations, secret abilities or resources, extreme deviations from doctrine or TTPs, or any other myriad "acts of the gods."

This thesis assumes that the resources used in the Blotto game are use-it-or-lose it. All entities must be assigned to fronts of a particular game. This is not to say that planners cannot assign reserve forces, but this may be a separate game for a separate objective. One study could be allocation of all forces across fronts of time (phases) or function - advance forces, main effort, supporting effort(s), follow-on forces, and reserve forces. The next study would be allocation of resources within those phases or functions. Regardless of what the particular study is, all forces considered in a particular study must be utilized.

Finally, this thesis assumes neither side employs chemical, biological, radiological, or nuclear weapons. The use of weapons of mass destruction significantly changes the strategic context of any other action or capability and an entirely different approach may be needed. This is not to say that the methods explored here may not be valuable in the assessment of weapons of mass destruction strategy, but this is an area needing further research and consideration.

1.6 Analysis Objectives

The objectives of the analysis presented in this thesis are to be proofs-of-concept for the limited but beneficial application of the General Blotto game theoretic framework to campaign phase planning. The intention is to understand the implications of the strategic allocation of resources across space, domains, or time. The General Blotto game is not meant to "solve for" a best COA, but to provide a tool for planners and commanders to understand and focus their options, identify decision points, and do the same for their adversary. This analysis shows that although the Colonel Blotto game has enjoyed limited use in military applications, the General Blotto game is a useful tool that deserves consideration.

1.7 Thesis Organization

Chapter II reviews the literature that provides relevant information to this study and supports the methodology of this thesis. Chapter III provides an in-depth discussion of the methodology used in this thesis. The General Blotto game is utilized to provide a framework for campaign planning and analysis. Chapter IV discusses the analysis of illustrative instances and a scenario involving this method and its results. Chapter V provides the conclusions of this research and suggestions for future application and research.

II. Literature Review

The best is the enemy of the good. By this I mean that a good plan violently executed now is better than a perfect plan next week.

- General George S. Patton[1]

2.1 Overview

The wars of the future will require more than a Joint mindset or synchronized Joint Force, but must additionally consider a force conducting interdependent operations [11]. Antiaccess / Area Denial (A2/AD) is widely understood as a strategy deterrence from a physical area [39]. Much has been written about the Chinese People's Liberation Army's development of long-range anti-ship ballistic missiles, island building, and layered approach to air defense systems [39]. Chinese colonels Liang and Xiansui [26] wrote in 1999 about tactics for developing countries such as China to compete in a high-tech war.

Countering this strategy is a task for a skilled Joint Force, operating in multiple domains, and assembled from the right components to apply the right pressure to the right points at the right time. Recently, Tangredi [40] has updated and refined his discussion of A2/AD to focus on a more nuanced understanding of Antiaccess which encompasses all conceivable domains - air, sea, land, space, cyber, and human, to include cultural, economic, and information itself. This requires considering Antiaccess as an all-encompassing strategy rather than just a campaign to be fought in the tactical battle space.

[1]Patton Jr., George S. *War as I Knew It.* Boston, MA: Houghton Mifflin, 1947:354. As a cavalry officer, Patton was accustomed to quick decision-making, speed in execution, and violence of action.

2.2 Framing the Research Topic

Antiaccess.

In a struggle between two militarily unmatched opponents, the weaker side must resort to strategies that limit direct contact, but instead capitalize on the weakening the will or the resolve of their stronger adversary [2]. This can be done overtly through terrorist activities, sowing fear throughout the populace through sporadic rear-echelon or homeland attacks [2], or it can be accomplished covertly through feints, utilizing geography or demography, or utilizing a time domain that is obviously non-advantageous to the greater power [2, 40].

Antiaccess refers to the ability to cordon off an area and control entry to it, effectively denying an adversary entry to the area. Area denial is to diminish, degrade, or destroy the adversary's freedom of movement or action within an area [40]. A2/AD generally refers to denying the freedom of action necessary to achieve military objectives. Tangredi argued that the two are different and should not be automatically bundled in the same acronym; that each mean different things, but the acronym is too loosely used to mean one or the other or both [40, 39]. This thesis uses each term, antiaccess, area denial, or A2/AD, to refer to that particular strategy.

An antiaccess strategy works to deny an adversary access to support areas, areas within striking range from where an attacker would traditionally launch attacks. For example, by building and arming islands surrounding the South China Sea, China is building a defensive antiaccess perimeter around the Sea. The point of these defenses is to keep adversaries (blue water navies) out of the Sea and out of strike distance of mainland China. China's long-range anti-ship missiles on the mainland and submarine forces are its area denial strategies for the South China Sea – they deny freedom of movement to any adversary within the area. Though the People's Liberation Army may not yet have global reach, they are a strong regional power that can effectively deter attacks on their homeland through their A2/AD strategy

Tangredi [40] offers five main characteristics of an antiaccess strategy:

1. the perception of the strategic superiority of an opponent,
2. the primacy of geography, as the element that most influences time and facilitates the combat attrition of the opponent's forces,
3. the general predominance of the maritime domain as conflict space,
4. the criticality of information and intelligence, and concomitantly the effects of strategic and operational deception, and
5. the determinative impact of extrinsic, sometimes apparently unrelated, events in other regions or globally.

It is important to note that he is a former U.S. Navy officer, so is most familiar with the maritime domain and is likely to frame such arguments in that context. However, given that the majority of the Earth is water and due to the open access granted to all nations on the seas, a focus on the maritime domain does make sense. Tangredi also notes [40] that successful antiaccess campaigns employ strategic deception. While operational and tactical deception are directed at an opponent's military operations, strategic deception is directed at decision making on the national command authority level and has a significant impact on the strategically superior power's decision making process. The potential for strategic deception will be an important consideration in creating and validating planning assumptions and evaluating beliefs.

The primary purpose of an advanced, capable standing army (military) is deterrence, not necessarily to actually fight [39]. The threat of being able to destroy an adversary if they stray too far outside international norms is the point of deterrence. A vital aspect of deterrence is not just the theoretical strength of the force, or how they "match up on paper," but one must make a periodic display of force – usually through training exercises or large-scale war games. Displaying capability goes a long way in backing up what one claims to have. Deterrence can be accomplished in three different ways: punishment, denial, and cooperation [18]. Deterrence by punishment is convincing an adversary that no matter how surprising or successful an attack may be, the counterattack will be devastating and too costly to consider the initial attack. Deterrence by denial focuses on defense, or having an effective counter to any attack an adversary might bring to bear. Deterrence through cooperation is achieved through mutual defense treaties, such as NATO. If an adversary understands that an attack on one is an attack on all, it changes the calculus of the potential strength of the actual adversary. Suddenly the punishment for an action becomes much more catastrophic.

A2/AD is a primary method for regional powers to exert their own deterrence from a global power and is an example of deterrence by denial. By denying support areas or even access to a domain, regional powers can do much to frustrate the freedom to operate of a much more powerful global power. The point is not to match head-to-head but be powerful enough to sufficiently raise the cost of taking action to a price the global power is unwilling to pay. Doing the same in a domain in which the regional power might other-wise have a potential first-strike advantage should have a powerful deterrent effect of its own.

In 1999, two Chinese Colonels, Qiao and Wang [26], outlined how a militarily inferior country such as China might counter the United States. Aside from the obvious covert methods touched upon earlier, such as: hacking into websites, targeting financial institutions, terrorism, using the media, and conducting urban warfare, the Colonels stated that weaker countries could essentially do anything: “the first rule of unrestricted warfare is that there are no rules, with nothing forbidden [26].” Seeing the world through a similar prism that Gerasimov [13] would write about 14 years later, they assert that,

> “strong countries make the rules while rising ones break them and exploit loopholes... The United States breaks [UN rules] and makes new ones when these rules do not suit [its purposes], but it has to observe its own rules or the whole world will not trust it [26, p. 2].”

Additionally, they note that future wars will be successful not through the skilled employment of an advanced and capable joint force, but through a new methodology that encompasses all aspects of a changing world, bringing every domain together in a common operating method [26].

Time.

Time is both an endogenous and exogenous parameter and is not only a factor in its obvious sense, but affects other qualities such as speed, surprise, flexibility, mobility, and can thus sometimes substitute for mass [29]. Control of time, the pace of the conflict, is attempted by both sides but at the same time, each side is subject to time. Although most think of time itself as a line (timeline), Einstein showed the relative nature of time [29]. Similarly, opponents and allies can both take advantage of, and fall victim to, the relative nature of time, especially when considering the psychological effects of time when manifested as waiting and surprise [29].

Planners must consider time in all contexts and account for its uncertain nature, both in the time it takes to complete an action or move a resource, and in the synchronization of resources in both space and time to be effective force multipliers. Assumptions can be made about time, but branches should be available should those assumptions not materialize as expected.

Integrating all available capabilities in all domains to optimize effects will overmatch the enemy through convergence. This is enabled through cross-domain synergy, multiple forms of attack, and the use of mission command and disciplined initiative [42]. The Joint Force currently converges capabilities through temporary synchronization of domain-specific solutions. Future operational success against a near-peer

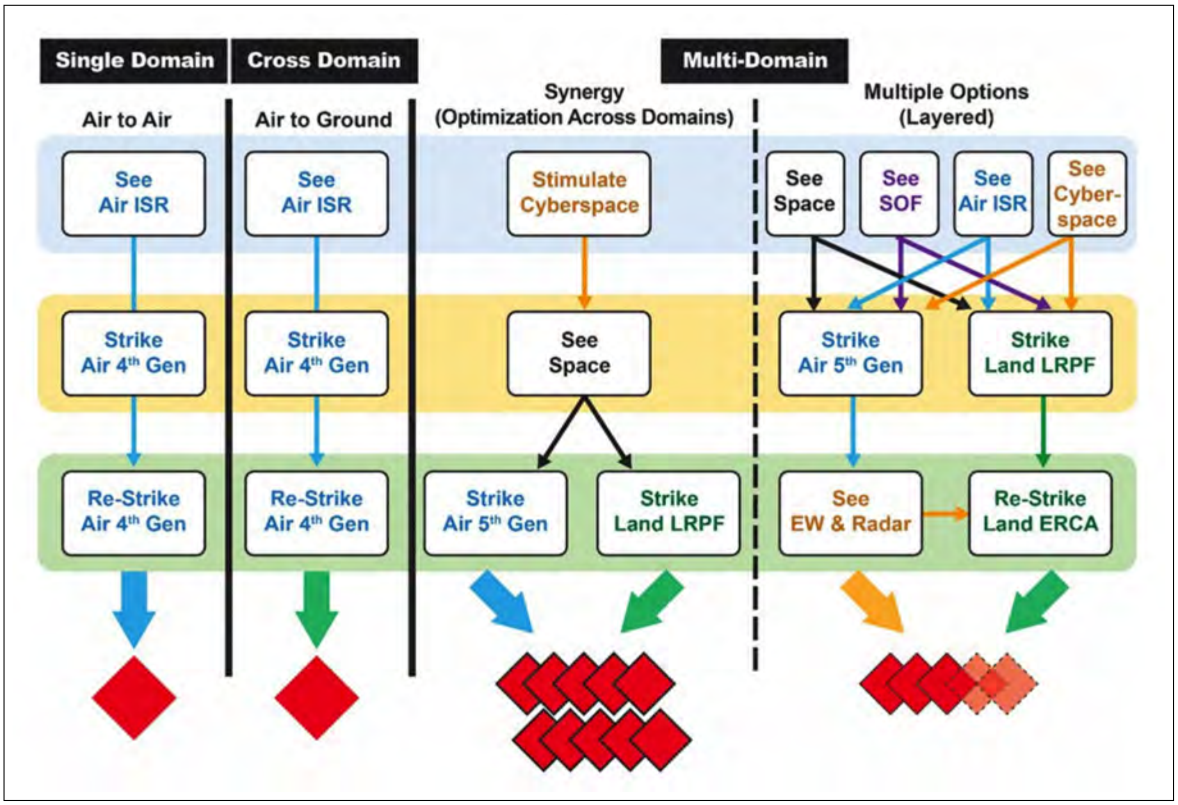


Figure 1. Converging capabilities to generate cross-domain synergy and layered options [42, p. 21].

adversary will require continuous and rapid integration of multi-domain capabilities to gain cross-domain overmatch at decisive times and spaces. Figure 1 shows how converging capabilities create cross-domain synergy. Operations will be won where capabilities can be synchronized in space and time to fully optimize their employments and create a marked advantage over an adversary. It will be imperative for planners to recognize dominant strategies that may be rapidly and effectively synchronized to take advantage of anticipated enemy situations.

Decision Support Systems.

A Decision Support System (DSS) utilizes a model to support managerial decision making in semi-structured or unstructured situations [41]. A DSS does not replace a decision maker, it aides in their decision making capabilities. The DSS uses data, provides a clear user interface, and can incorporate the decision maker's own insights. This thesis provides a framework for creating a DSS for planners and commanders to explore COAs and gain insight on a potential adversary's actions and their effect on concurrent friendly actions. It is important that a DSS offers a means to input data, assembles or models that data to give it meaning, provides analysis capabilities, and creates an output that is meaningful to the user.

Multi-Domain Operations.

Multi-domain operations are an emerging doctrinal concept. Although this thesis focuses on Joint Doctrine, the recently published U.S. Army Training and Doctrine Command Pamphlet 525-3-1 [42] contains the most recent guidance on this emerging doctrine. As the Air Force and the Army are the leading proponents formulating joint doctrine on multi-domain operations, this is an appropriate published reference. Figure 2 [42, p. 9] provides an overview of competition and armed conflict, threat considerations, and the domains considered, and capabilities utilized in the future.

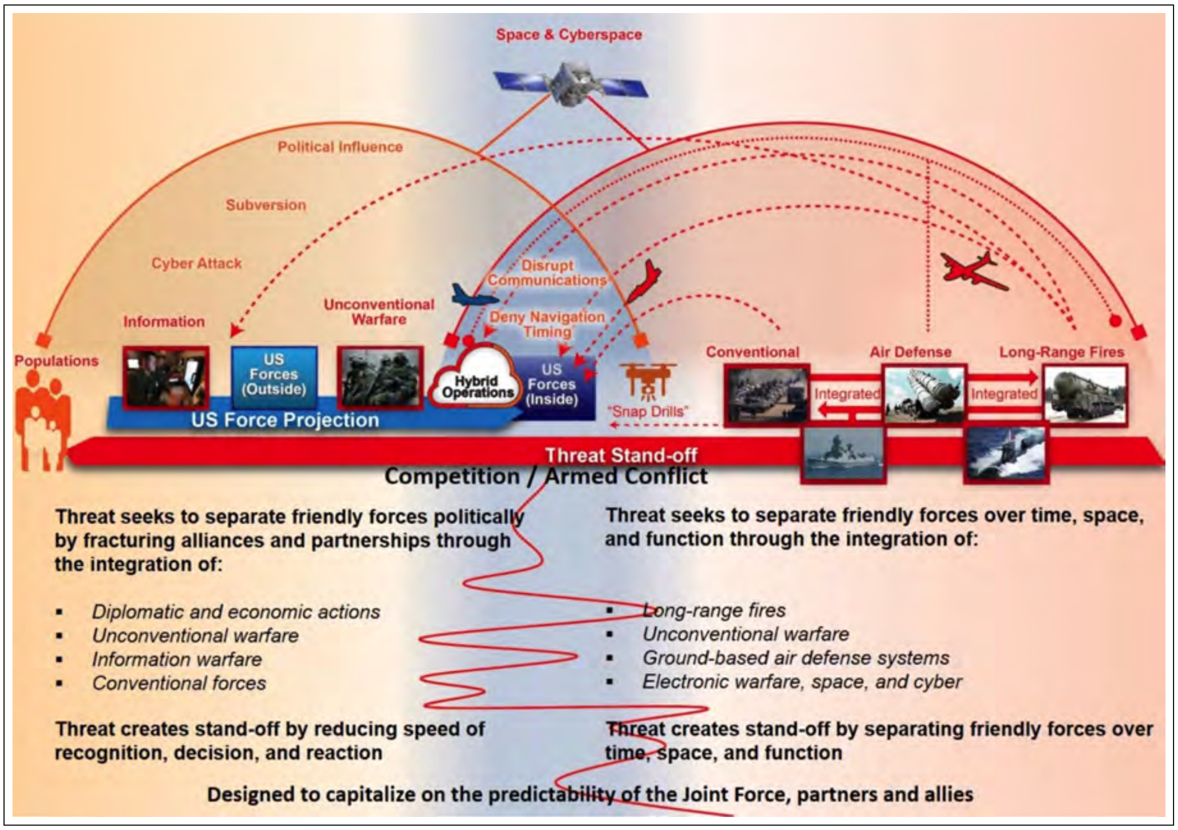


Figure 2. Competition and Armed Conflict [42, p. 9].

As adversaries such as China and Russia have become more powerful and invested in new technologies and expanded their presence in newer domains such as space and cyberspace, the potential "global battlefield" has been expanded in time (as a blurred distinction between peace and war), in domains, and in geography (long-range kinetic weapons, deep strike capabilities, and cyber attacks) to create tactical, operational, and strategic stand-off [42, p. vi].

China and Russia have recently invested in and developed a systematic approach to counter existing joint doctrinal approaches to the conduct of war. The resulting A2/AD strategies create strategic and operational stand-off that separates the elements of the Joint Force in time, space, and function. To overcome these problems presented by Chinese and Russian operations in competition and conflict, the Joint Force must apply three interrelated tenets: calibrated force posture, multi-domain formations, and convergence [42, p. vii]. Calibrated force posture considers the force's position in space and its ability to maneuver across strategic distances. Multi-domain operations will occur across multiple domains in contested spaces against a near-peer adversary. These three tenets are mutually reinforcing and common to all multi-domain operations, but how they are balanced and emphasized across domains will vary with each operation.

2.3 Game Theory

Game theory is the study of mathematical models of strategic interaction between rational players [31]. Because the players are assumed to be rational, game theoretic models can be evaluated mathematically based on their expected behavior concerning probable outcomes. If a player acts irrationally, i.e., makes a decision that is not in their best interest based on the expected outcome of an action, evaluation of that action becomes much more difficult. Game theory can provide great insight to interactions between rational, strategic players, but care must be taken to ensure that both players are acting rationally or are basing their decisions on known assumptions or motivations. Knowing the true underlying assumptions of an adversary is important in effectively predicting (and modeling) how they may act. This thesis will apply a game theoretic framework to exploring strategic decision making and planning while keeping in mind that there is uncertainty in assuming the rationality of an adversary (or properly understanding the rationale of an adversary's decision-making).

Game theory began in earnest in the early 1900's but was collected and published as a coherent theory by von Neumann and Morgenstern [33] in 1944. This work set the stage for an ever-expanding body of knowledge that continues through today. This thesis is concerned with one game typified as one-shot, two-player, simultaneous, zero-sum, non-cooperative, asymmetric, and is modeled assuming complete information. However, it follows that the completeness of the information is based on the validations of utilized assumptions to further understand and analyze the game.

Colonel Blotto.

The Colonel Blotto game is a fundamental strategic resource allocation model in multiple dimensions. When considering how to allocate resources across multiple dimensions with the intent of defeating an opponent who is simultaneously doing the same, the Colonel Blotto game is a good place to start.

In 1921, Borel [5] introduced the Colonel Blotto game, dividing resources among a number of fronts with the goal of winning a majority of the fronts by having more resources than the opponent. His motivation was to see if there was a single, superior strategy given that two players who use the same strategies will have equal chance of success. Borel's model was for three fronts. Gross and Wagner [16] extended this to situations of more than three fronts in 1950 as well as introducing the name of the commander of troops, Colonel Blotto.

For consistency in the introduction of the Colonel Blotto game, the same notation as Golman and Page [14] is utilized. The Colonel Blotto game is a zero-sum game of strategic mismatch between two players, X and Y. A strategy for Player X can be written as a real vector of components, x_i^{CB}: $(x_1, x_2, \ldots, x_m)$ with

$$\sum_{i=1}^{m} x_i^{CB} = 1, x_i^{CB} \in [0, 1], \tag{1}$$

where x_i is the fraction of the budget allocation to front i, and m is the total number of fronts. Likewise, a strategy for player Y can be written as a real vector of components, y_i^{CB}: $(y_1, y_2, \ldots, y_m)$ with

$$\sum_{i=1}^{m} y_i^{CB} = 1, y_i^{CB} \in [0, 1]. \tag{2}$$

Table 1 shows the various pure strategies available to Colonel Blotto to distribute five resources across three fronts.

Scoring (determining a winner) per Golman and Page [14], occurs by determining which player has won a majority of the fronts and then what their payoff is for each of the won fronts. The payoff to X against Y is:

$$\sum_{i=1}^{m} \operatorname{sgn}(x_i^{CB} - y_i^{CB}), \text{ where } \operatorname{sgn}(\chi) := \begin{cases} 1 & \text{if } \chi > 0, \\ 0 & \text{if } \chi = 0, \\ -1 & \text{if } \chi < 0. \end{cases} \tag{3}$$

Table 2 shows an example of a two-player Colonel Blotto game with five resources on three fronts. Table 3 shows an example of a two-player Colonel Blotto game with five resources on three fronts.

Much of the literature [1, 8, 30, 37] is focused on “solving” Colonel Blotto and finding optimal strategies. By running experiments, simulations, and actual tournaments, much time has been invested in studying the evolution of a Colonel Blotto

Table 1. All strategies for a simple two-player Colonel Blotto game with five resources on three fronts

Strategy	Front 1	Front 2	Front 3
S1	5	0	0
S2	0	5	0
S3	0	0	5
S4	4	1	0
S5	4	0	1
S6	1	4	0
S7	1	0	4
S8	0	4	1
S9	0	1	4
S10	3	2	0
S11	3	0	2
S12	2	3	0
S13	2	0	3
S14	0	3	2
S15	0	2	3
S16	3	1	1
S17	1	3	1
S18	1	1	3
S19	2	2	1
S20	2	1	2
S21	1	2	2

Table 2. Simple two-player Colonel Blotto game with five resources on three fronts where it is assumed the defender wins ties

Player	Front 1	Front 2	Front 3
Defender	2	2	1
Attacker	3	2	0
Winner	Attacker	Defender	Defender

Table 3. Simple two-player Colonel Blotto game with five resources on three fronts where it is assumed the defender wins ties

Player	Front 1	Front 2	Front 3
Defender	2	2	1
Attacker	3	0	2
Winner	Attacker	Defender	Attacker

strategy. This is where the utility of the military applicability of the game is removed from that found in earlier studies [38, 17]. Blotto games are popular and very useful and well understood in electoral politics, especially in national elections in the United States because strategies are laid out over many fronts (50 States) and elections are partially reproducible (they happen over time, they are common, electorates can be studies, results can be analyzed, polls can be conducted). In planning for a military strategy, the plan is for a one-time event, and the opposition's state is not perfectly known. In addition, there are essentially no rules in war, so it is hard to model as a game. By focusing on the insight that the Blotto game can bring to military planning, its utility can be discovered and appreciated. As Geoffrion stated, "the purpose of mathematical programming is insight, not numbers [12]."

Military Conflicts.

A game is symmetric if both players have the same strategy set [31, p. 97]. Even if they start with the same number of resources, if the strategy sets differ (subject to different construction constraints) then the game is asymmetric. This thesis deals with an asymmetric game, where each player has their own, distinct strategy set, as different militaries tend to do.

The Colonel Blotto game is especially important in understanding asymmetric conflict. Arreguín-Toft [3] analyzed over 200 asymmetric conflicts since 1800 and found that the stronger actor prevailed 72% of the time. This is significant since he defined a conflict as asymmetric if one side was stronger in terms of forces and population by a factor of ten or more. Upon further analysis, he also found that over these 200-plus years, the weaker side has been winning at a higher and higher rate - from 12% in the first 50 years, to nearly 50% in the last 50 years. Weaker forces are learning how to win asymmetric wars, but how are they doing this? Arreguín-Toft noted that when the mis-matched opponents fought head-to-head, the stronger force won over 80% of the time. However, when the weaker force employed guerrilla tactics, i.e. only fighting the stronger force when there is some clear advantage, they won over 60% of the time. By adding non-traditional fronts (increasing n), they are more likely to wear the stronger side down and win the war. Another interesting revelation was that nearly 80% of the losers of asymmetric conflicts never changed their strategies.

For a great power, there is much to learn from Arreguín-Toft's analysis. Just as the weak power will want to expand the fronts, the great power will want to limit the fronts (fight them over there instead of fighting them here). The weaker power will want to expand the timeframe while the stronger side should consolidate the timeframe and win decisively (strive for the high payoff). Finally, the great power should be willing to change strategies. To do this, the great power must be adaptable to change and have considered alternate COAs and must be prepared to identify how and when to change. These are all themes which will be visited later in this thesis when considering model creation and analysis.

In 1954, Haywood [20] recounted two World War II battles as "theory of games" as a way to analyze decision making. The Rabaul-Lae Convoy Situation and the Avranches-Gap Situation are famous examples of military decision making in game theory. Haywood applied Von Neumann and Morgenstern's two-player, zero-sum game to the Rabaul-Lae Convoy Situation, or the "Battle of the Bismarck Sea." The Battle of the Bismarck Sea (2–4 March 1943) near Lae, New Guinea involved a Japanese troop and supply convoy heading from Rabaul to Lae via either a Northern Route or a Southern Route. The Allies could send reconnaissance aircraft either to the north, with bad visibility, or south as shown in Figure 3. Table 4 shows the two strategies to be considered by the Allies and the Japanese. If reconnaissance aircraft were to be sent north and the fleet sailed north, two days of bombing of the fleet could be expected due to the effects of the bad weather. If the aircraft went north and the fleet went south, then the fleet would still be bombed for two days. There

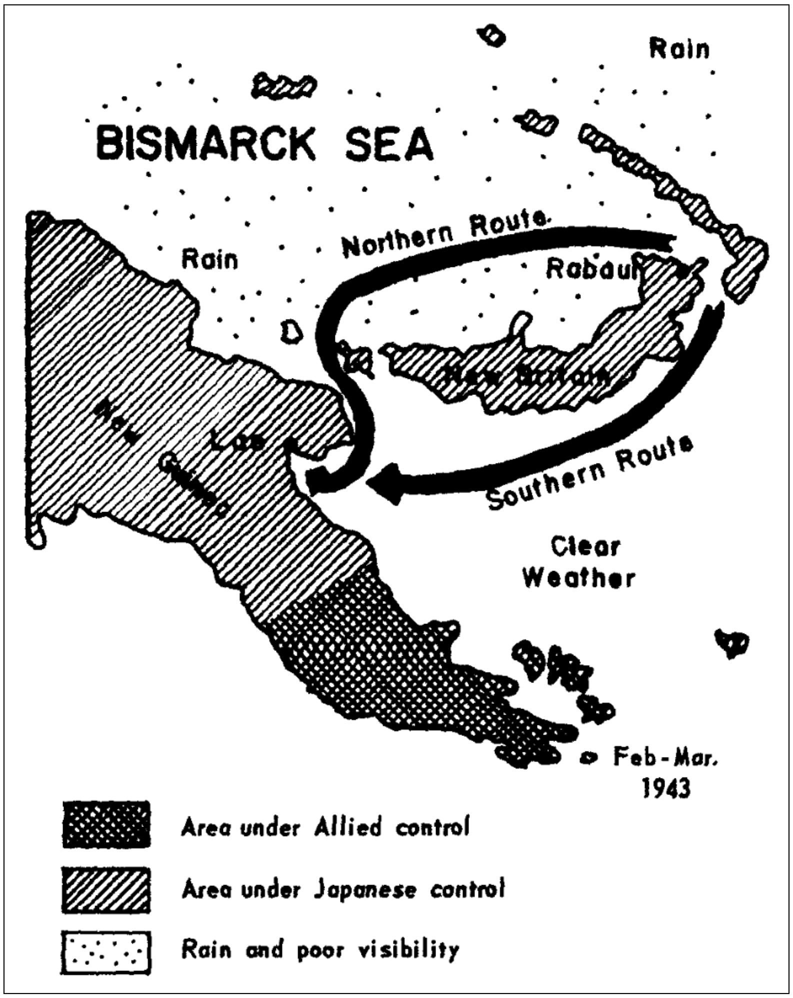


Figure 3. The Rabaul-Lae Convoy Situation. The problem is the distribution of reconnaissance to locate a convoy with may sail by either one of two routes [20, p. 366].

Table 4. The Battle of the Bismarck Sea payoff matrix. Expected days of bombing given strategies chosen [20, p. 370].

		Japanese Strategies		Minimum of row
		North	South	
Allied Strategies	North	2 days	2 days	2 days (maximin)
	South	1 day	3 days	1 day
Maximum of column		2 days (minimax)	3 days	

would be a delay in commencement due to the time required to determine that the fleet had sailed south, but the good weather would allow expedited and continued bombing effects. If the reconnaissance aircraft flew south and the fleet sailed north, only one day of bombing could occur, due to the late initiation and bad weather. If the aircraft flew south and the fleet sailed south, then a full three days of bombing would commence. The model suggested that the optimal Allied strategy would be to send aircraft north and for the Japanese fleet to sail north, which was what actually happened in 1943.

Table 5. Novikov's models of warfare table [34, Table 1. p. 1742].

Hierarchical level	Modeled phenomena/processes	Modeling tools
5	Spatial distribution of forces and means	The colonel Blotto game and its modifications
4	Temporal distribution of forces and means	Optimal control, repeated games, etc.
3	Size dynamics	Lanchester's equations and their modifications
2	"Local" interaction of units	Markov models
1	Interaction of separate military units	Simulation, the Monte Carlo method

Novikov explored trends in the design of complex models of warfare [34]. He considered canonical models, namely Lanchester's models and Colonel Blotto games, and considered their usefulness in a hierarchical model of warfare. Table 5 is Novikov's [34] hierarchical model of warfare which acknowledge the utility for Blotto games in high-level planning with other methods more appropriate for modeling other levels of warfare. He found that the Colonel Blotto model is appropriate for, "spacial distribution of forces and means."

The Utility Function and Payoffs.

One simple way of showing a game is by using a game matrix or "normal form." This is really a table of utility [31]. In this thesis, utility is the relative gain or satisfaction a player gets from a particular outcome, or payoff. To create a game matrix, first work out the utility values. Low utility values or payoffs are least attractive to a player and high utility values or payoffs are more attractive to a player.

The expected utility is the average utility a player might see over the distribution of the other player playing their strategy set. The expected utility is an important consideration in evaluating which strategies are attractive or not. A note of caution is to remember that expected utility is an expectation over time or given a random distribution of strategies actually played. In some cases, especially when a game is one-shot, or a single occurrence, additional input in the form of expert opinion may be warranted to help evaluate what strategy is most appropriate.

Payoffs.

Table 6 [35, p. 199] shows the payoff table for the game, rock - paper - scissors. Reading left to right, rock ties rock and gets a payoff of zero, rock loses to paper and gets a payoff of minus one, and rock beats scissors to get a payoff of plus one. Each subsequent row performs in a similar manner.

Table 6. Rock - paper - scissors payoff table.

	Rock	Paper	Scissors
Rock	0	-1	1
Paper	1	0	-1
Scissors	-1	1	0

Figure 4 [24] shows the more complex version of rock - paper - scissors - rock - paper - scissors - spock - lizard, first developed by Kass and Bryla. By expanding the number of game options, it becomes harder to detect patterns in an opponents strategy. Table 7 is the payoff matrix for rock - paper - scissors - spock - lizard.

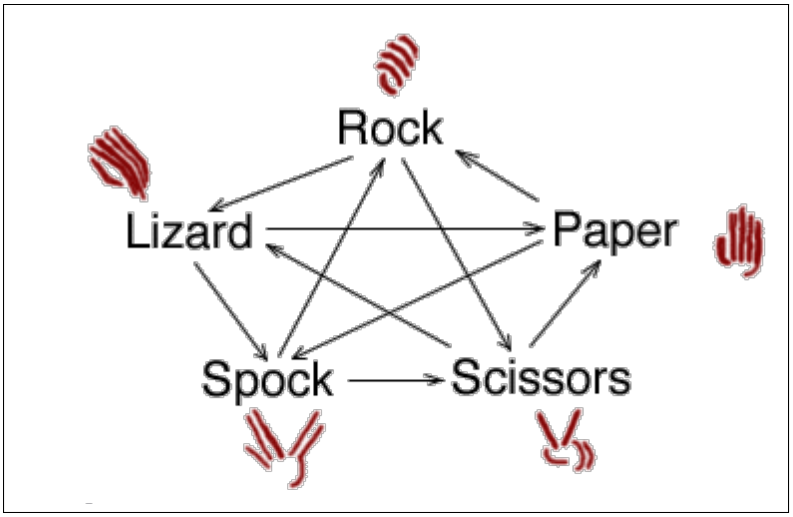


Figure 4. Rock - Paper - Scissors - Spock - Lizard [24]. In this game, Scissors cuts Paper covers Rock crushes Lizard poisons Spock smashes Scissors decapitates Lizard eats Paper disproves Spock vaporizes Rock crushes Scissors.

Table 7. Rock - paper - scissors - spock - lizard payoff table.

	Rock	Paper	Scissors	Spock	Lizard
Rock	0	-1	1	-1	1
Paper	1	0	-1	1	-1
Scissors	-1	1	0	-1	1
Spock	1	-1	1	0	-1
Lizard	-1	1	-1	1	0

Table 8 is the payoff matrix for the three-front, five-resource Blotto game strategies outlined in Table 1 where it is assumed the defender wins ties - therefore winning pays 1 and losing pays -1. Given the mixed strategies outlined by Gross and Wagner [16] and Hart [19], only those strategies that fall on [0 ,3], strategies S10-S21, are included.

The payoff matrix in Table 8 portrays payoffs from the point of view of the Defender. If an attacking planner were to utilize this, they would be looking for “-1’s,” or the strategy matchups in which the Defender loses. The best options come from S10-S15, as each of these strategies beats three of the Defender’s strategies, while S16-S18 each beat only two, and S19-S21 only beat one. It makes sense that strate-

Table 8. Payoff matrix for Blotto game with strategies from Table 1 where Defender wins stalemates (i.e. Attacker does not win).

	Attacker											
Defender	S10	S11	S12	S13	S14	S15	S16	S17	S18	S19	S20	S21
S10	1	1	1	1	**-1**	1	1	**-1**	1	1	1	1
S11	1	1	1	1	1	**-1**	1	1	**-1**	1	1	1
S12	1	**-1**	1	1	1	1	**-1**	1	1	1	1	1
S13	**-1**	1	1	1	1	1	**-1**	1	1	1	1	1
S14	1	1	1	**-1**	1	1	1	1	**-1**	1	1	1
S15	1	1	**-1**	1	1	1	1	**-1**	1	1	1	1
S16	1	1	1	1	**-1**	**-1**	1	1	1	1	1	**-1**
S17	1	**-1**	1	**-1**	1	1	1	1	1	1	**-1**	1
S18	**-1**	1	**-1**	1	1	1	1	1	1	**-1**	1	1
S19	1	**-1**	1	1	**-1**	1	1	1	1	1	1	1
S20	**-1**	1	1	1	1	**-1**	1	1	1	1	1	1
S21	1	1	**-1**	**-1**	1	1	1	1	1	1	1	1

gies S10-S15 would be the most beneficial for the Attacker, as these strategies massed attacking forces on two of three fronts required to win the war.

The payoff matrix in Table 9 portrays payoffs from the point of view of the Defender when ties are allowed where neither side wins and the payoff is zero. If an attacking planner were to utilize this, they would be looking for "-1's," or the strategy matchups in which the Defender loses. The best options come from S10-S15, as each of these strategies beats three of the Defender's strategies, while S16-S18 each beat only two, and S19-S21 only beat one. It makes sense that strategies S10-S15 would be the most beneficial for the Attacker, as these strategies massed attacking forces on two of three fronts required to win the war.

Nash Equilibria.

The Nash equilibrium is a profile of strategies from which no player can benefit from a unilateral deviation [32]. If each player chooses a strategy and neither player can improve their situation by changing their strategy, then it is a Nash equilibrium. A general reduction can be made for Colonel Blotto, General Blotto, and other dueling

Table 9. Payoff matrix for Blotto game with strategies from Table 1 allowing for stalemates of payoff zero.

	Attacker											
Defender	S10	S11	S12	S13	S14	S15	S16	S17	S18	S19	S20	S21
S10	0	0	0	1	-1	0	0	-1	1	0	1	0
S11	0	0	1	0	0	-1	0	1	-1	1	0	0
S12	0	-1	0	0	0	1	-1	0	1	0	0	1
S13	-1	0	0	0	1	0	-1	1	0	0	0	1
S14	1	0	0	-1	0	0	1	0	-1	1	0	0
S15	0	1	-1	0	0	0	1	-1	0	0	1	0
S16	0	0	1	1	-1	-1	0	0	0	0	0	-1
S17	1	-1	0	-1	0	1	0	0	0	0	-1	0
S18	-1	1	-1	0	1	0	0	0	0	-1	0	0
S19	0	-1	0	0	-1	0	0	0	1	0	0	0
S20	-1	0	0	0	0	-1	0	1	0	0	0	0
S21	0	0	-1	-1	0	0	1	0	0	0	0	0

games to find a Nash equilibrium [1].

A pure strategy describes how a player will play a game - it describes the move a player will make given any situation. Therefore, the strategy set is the set of pure strategies that a player has. A mixed strategy is an assignment of a probability to a pure strategy. A pure strategy is in the support of a mixed strategy if that pure strategy is played with positive probability according to the mixed strategy. Nash's Theorem states that there must be at least one Nash Equilibrium for all finite games. If no equilibrium exists in pure strategies, one must exist in mixed strategies. A mixed strategy is a probability distribution over two or more pure strategies. That is, the players choose randomly among their options in equilibrium and if the mixtures are mutual best responses, then the set of strategies is a mixed strategy Nash equilibrium. If the players are rational and the pure strategies they are playing are mutual knowledge, then those strategies must form a Nash equilibrium [31, p.98]

In 1950, Gross and Wagner [16] showed that the Colonel Blotto game has a mixed strategy equilibrium in which the marginal distributions are uniform on $[0, \frac{2}{m}]$ for all the m fronts. Since all allocations between 0 and $\frac{2}{m}$ resources are equally likely, the opponent has no preference for how he or she should play in response, provided no more than $\frac{2}{m}$ resources are allocated. Hart [19] built on this and found that for n discrete indivisible units to be allocated between m fronts, the mixed strategies were on $[0, \frac{2n}{m}]$. Given all of the available plays shown in Table 1, the mixed strategies that fall on $[0, \frac{2n}{m}]$ are $[0, \frac{10}{3}]$, or $[0, 3.\bar{3}]$, rounded to [0, 3]. These were the strategies utilized in Table 2 and Table 3.

Mixed strategies are a concise description of what might happen in repeated play. They are a count of pure strategies in the limit. Mixed strategies help to describe the expectation of two agents chosen from a population, all having deterministic strategies. A mixed strategy is the probability of getting an agent who will play one pure strategy or another. Players randomize their strategies when they are uncertain about the other player's action. In repeated games, players may devolve into predictable variations of their strategies. Playing a truly randomized strategy will help to confuse one's opponent (such as in rock-scissors-paper).

Aumann and Brandenburger [4] observed that players' choices will constitute a Nash equilibrium if each player is rational, they know their own payoff function, and they know the strategy choices of the other player. This thesis assumes that these conditions are true for the sake of creating the model, but acknowledges that one or more may not be true in reality. If the apparent rationality of the adversary is in question, expert belief can help bound the feature space and narrow the strategy choices.

Ahmadinejad, Dehghani, Hajiaghayi, Lucier, Mahini, and Seddighin [1] studied the problem of computing Nash equilibria in zero-sum games. They found that as the size of the strategy space increases, standard methods for computing equilibria of zero-sum games fail to be computationally feasible [1]. This thesis considers mixed strategies when feasible, but the ultimate goal of the framework is to focus on narrowing the feature space based on facts and validating assumptions so that it may be narrowed to a size that might be easily analyzed. The ultimate goal is to identify feasible COAs for consideration and gain insight as to what strategies are feasible, not to necessarily "predict" and optimal COA (COA evaluation and decision will come later in the JPP).

Linear Programming.

Classical game theory finds optimal strategies utilizing the minimax theorem [31, p. 89]. Assuming that in any two-person cost matrix in a zero-sum game (there is a winner and a loser, competition does not make both sides better or worse), there is a strategy for each player such that neither player can improve their expected payoff by adopting a different attack or defense. An example of the minimax theorem is illustrated in Table 4. The Allied commander looks to maximize the minimum number of days of bombing, while the Japanese fleet commander wanted to minimize the maximum number of days his fleet would be under attack. In that example, both commanders' optimal strategies likely resulted in two days of bombing.

General Blotto.

Kovenock [25] examined a two-stage model of asymmetric conflict based on the classic Colonel Blotto game in which players have the ability to increase the number of battlefields contested. He showed that this generalized example endogenized the "dimensionality" of conflict. He found that "In equilibrium, if the asymmetry in the players' resource endowments exceeds a threshold, the weak player chooses to add battlefields, while the strong player never does. Adding battlefields spreads the strong player's forces more thinly, increasing the incidence of favorable strategic mismatches for the weak player [25]."

The General Blotto Problem was introduced by Hart [19] in 2007 as a class of integer-valued allocation games that expands on the original Colonel Blotto game by replacing specifically chosen resource levels with a chosen distribution. Golman and Page [14] outlined the General Blotto game that allows for non-winner-takes-all payoffs, externalities between fronts, and pairwise competition among a population of players. Golman and Page generalize the Colonel Blotto game, invoking a witticism in the process and promoting Colonel Blotto to General. The class of "General Blotto" games allows for different valuations of objectives. The payoffs of different strategies depend not only on the number of objectives captured, but on the valuation of the objectives as well [14].

General Blotto [14] better aligns with a modern military allocation game since Colonel Blotto is a two-player game allocating resources to fronts where sheer mass wins the day. General Blotto allows for non-winner-takes-all payoffs, externalities between fronts, and pairwise competition among a population of players, and therefore more accurately captures these real-world situations [14, p.282]. The Colonel Blotto game is fairly straightforward, but modern conflict is much more complicated than similar forces amassed on a battlefield and then fighting head-to-head. Today, as in the past, a military power must decide where to allocate its resources. But, a modern military also cares about resources at combinations of fronts. The emphasis on the number of troops on a field has been replaced with a concern over the number of troops in a theater of operation or resources given to a particular *domain*. A military also allocates resources to different departments and projects. Fronts may be strategic goals or civil considerations rather than literal geographic areas [14, p.282].

Experiments.

Experimental studies of Colonel Blotto include Chowdhury, Subhasish, Kovenock, and Sheremeta [8] and Modzelewski, Stein, and Yu [30]. The first show that under a lottery treatment, the equilibrium prediction is that each player should divide their resources equally across all fronts. The experimental results support this prediction. Moreover, deviations from equilibrium behavior result in lower payoffs. Under the auction treatment, equilibrium requires that the disadvantaged player stochastically allocates zero resources to a subset of fronts and the advantaged player allocates random, but positive, resource levels across the fronts. Again, the data support this theoretical prediction and deviations from equilibrium behavior in the form of strategies exhibiting low dispersion of allocations across fronts at a point in time or within a front over time are associated with lower payoffs. Winning a box in a period encourages the subject to allocate more resources to that box in the next period There is a clear trend that people mix more evenly as they play more games against a single opponent. This implies that the dominant learning effect is that people learn how their opponent plays rather than how the set of their opponents play in aggregate, to the point that people default back to different strategies when playing a new opponent for the first time.

> "Humans have consistent biases in many of these games, such as rock-paper-scissors, and consequently, these biases can be exploited in manners that theoretical studies never predicted. The Colonel Blotto class of games has many applications to real world problems including business investment under competition and strategy computer games. There is a great deal of work left for illustrating and predicting human biases and deciding optimal play in games where humans are involved [30, p.17]."

Although many applications and extensions of Colonel Blotto were studied, there was a lull in significant developments until Roberson in 2006 [37] . Roberson established new and novel solutions that do not utilize the regular $n-$gons of Borel [5] or Gross and Wagner [16, 15]. Roberson applied the theory of copulas, the functions that map univariate marginal distributions into joint distributions, to extend "the literature on the Colonel Blotto game by characterizing the unique equilibrium payoffs for all symmetric and asymmetric configurations of the players' aggregate levels of force and characterizing the complete set of equilibrium univariate marginal distributions for most of these configurations [37, p.19].”

Most Blotto discussions/research focus on identifying strategy over time. Although this could be useful in a long term study based on historical data, it would still be difficult to enumerate all aspects and understand all interactions.

2.4 Joint Doctrine and Planning Considerations

Campaign Planning.

Joint Publication (JP) 5-0 defines a campaign [22, p. xviii] as a series of related military operations aimed at accomplishing strategic and operational objectives within a given time and space. Given what is known of an adversary’s use of antiaccess strategies and the need for and complexity of multi-domain operations, planning a campaign can get very complicated, very quickly. There are many different factors to evaluate, from the physical and virtual terrain, to friendly readiness and posture, and adversarial strategy and capability. To plan a campaign, the Joint Force utilizes the JPP to systematically attack the problem.

Figure 5 from JP 5-0 [22, p. V-3] shows the steps of the JPP horizontally across the top of the figure. This thesis is concerned primarily with:

- the Joint Intelligence Preparation of the Operational Environment (JIPOE),
- Mission Analysis,
- COA Development, and
- COA Analysis.

The focus is on brainstorming the ranges of feasible friendly and enemy COAs and establishing a framework in which to narrow those options to those which will be most beneficial to study in detail and consider as published plans.

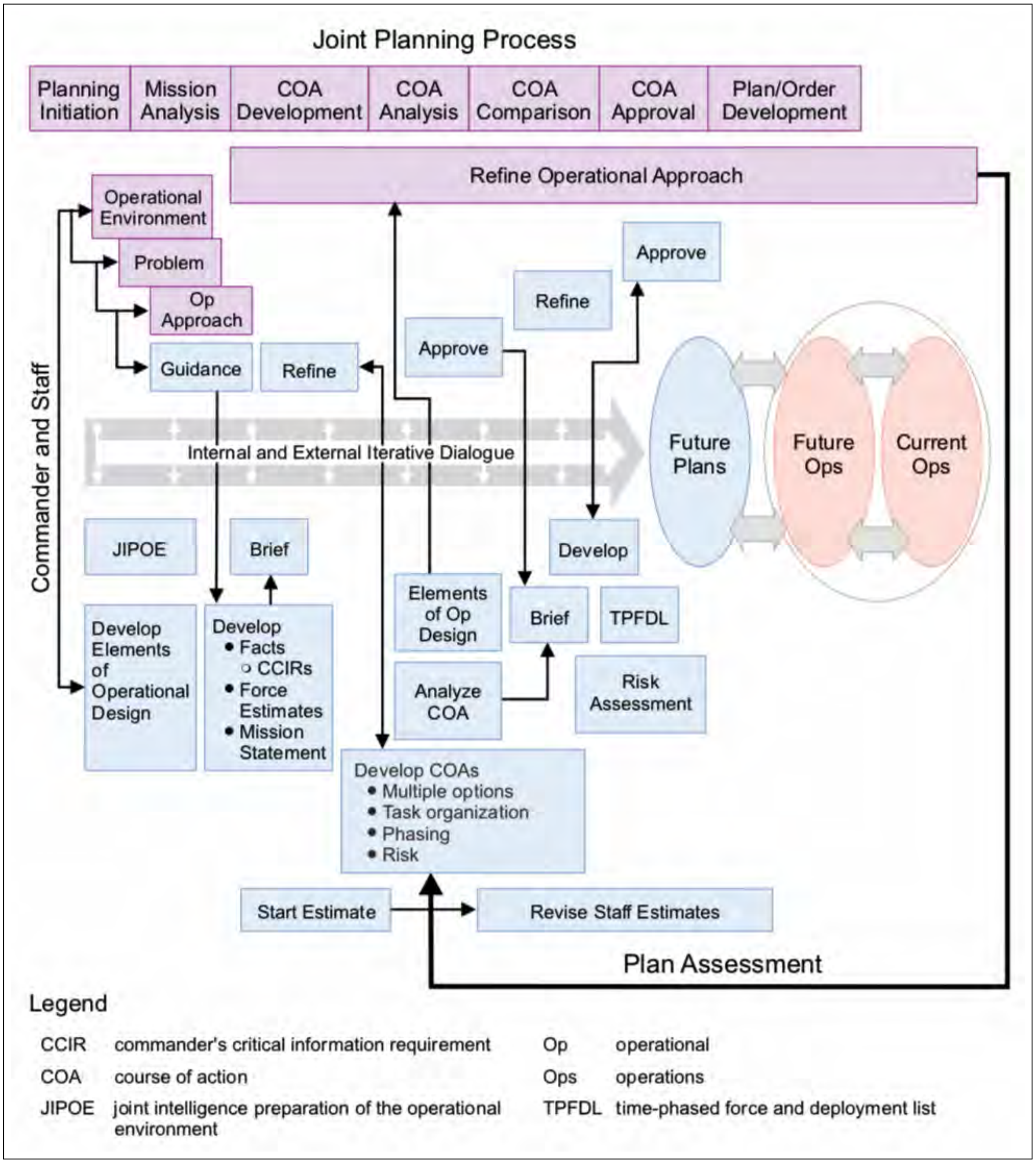


Figure 5. Overview of the Joint Planning Process [22, p. V-3].

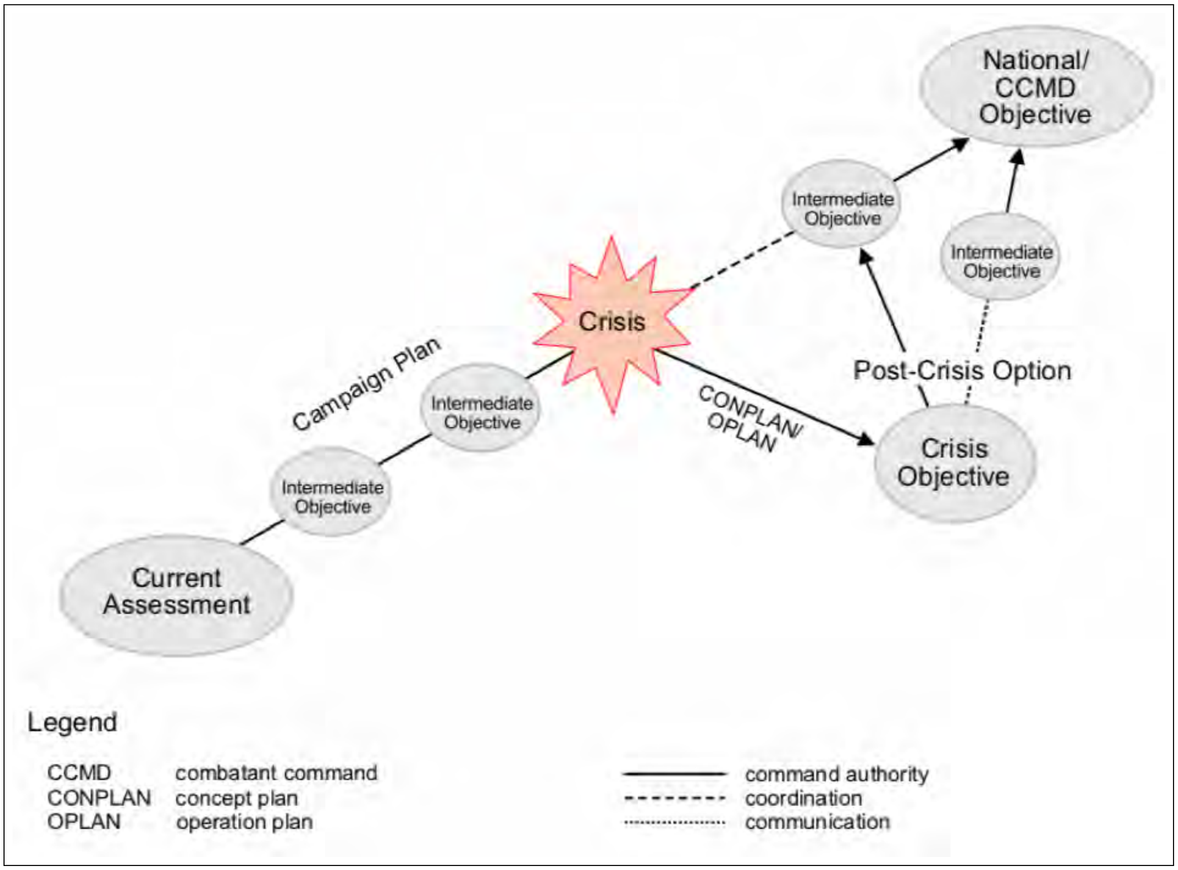


Figure 6. The typical trajectory of the campaign [22, p. III-3].

Figure 6 from JP 5-0 [22, p. III-3] shows how operation plans (OPLANs) can break from an overall campaign plan and depending on the outcome, still meet the overall objectives. This is why it is important for planners and commanders to understand multiple available strategies or plans and consider strategic agility in addressing developing situations.

Campaign Phases.

By utilizing the phasing construct, planners can divide the campaign to be able to provide opportunities to synchronize in time a space [22, p. xxiii]. A phase can be characterized by the focus that is placed on it. Phases are distinct in time, space, and/or purpose from one another, but must be planned in support of each other and should represent a natural progression and subdivision of the campaign or operation. Each phase should have a set of starting conditions that define the start of the phase and ending conditions that define the end of the phase. The ending conditions of one phase are the starting conditions for the next phase.

Figure 7 from JP 5-0 [22, p. V-13] shows how six notional phases based on general groups of activities provide a basis for thinking about a joint operation. By utilizing the phasing structure to group activities, joint force commanders and staffs can more easily visualize, plan, and execute the entire operation. They can define their requirements in terms of forces, resources, time, space, and purpose to achieve objectives.

Figure 8 from JP 5-0 [22, p. V-15] shows how combat missions and tasks can vary widely depending on context of the operation and the objective. Broken down by the same Phase I-V used in Figure 7, most combat operations will require the commander to balance offensive, defensive, and stability activities. This is particularly evident in a campaign or major operation, where combat can occur during several phases and stability activities may occur throughout.

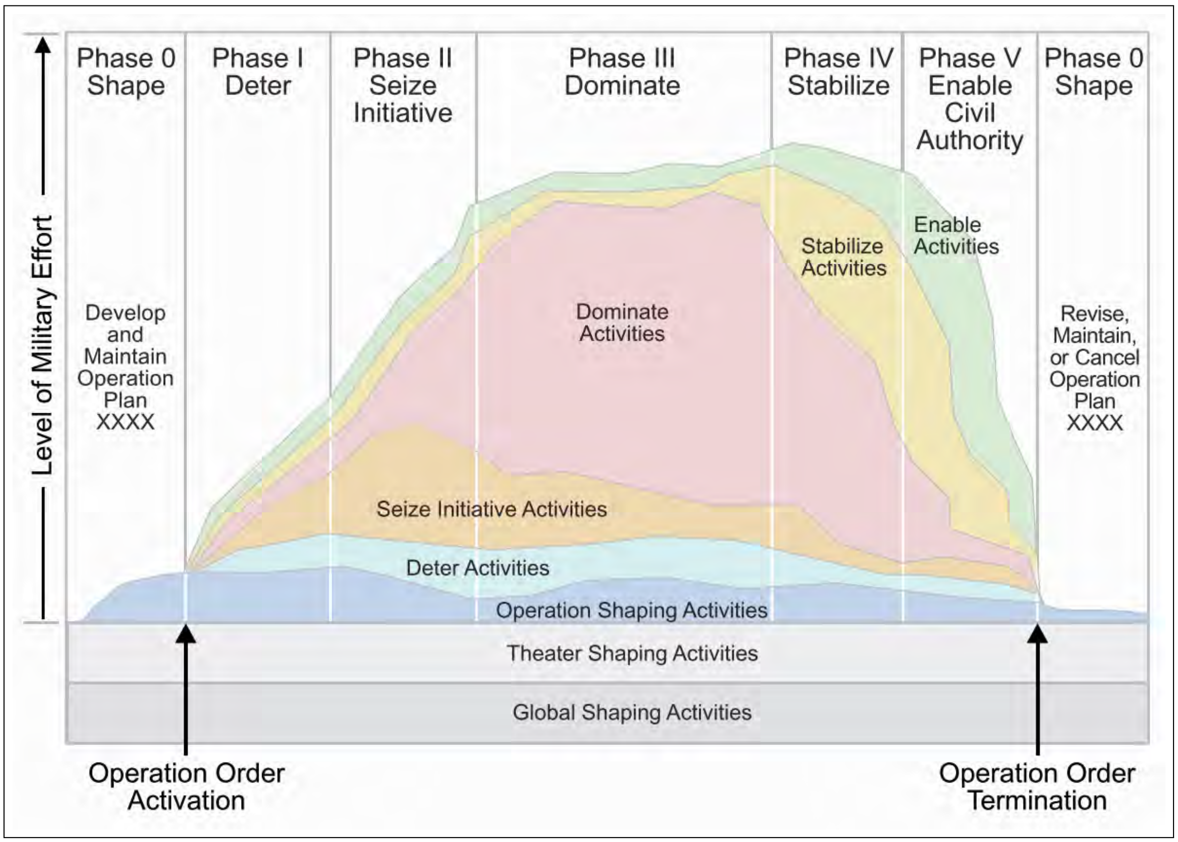


Figure 7. Phasing an Operation Based on Predominant Military Activities [22, p. V-13].

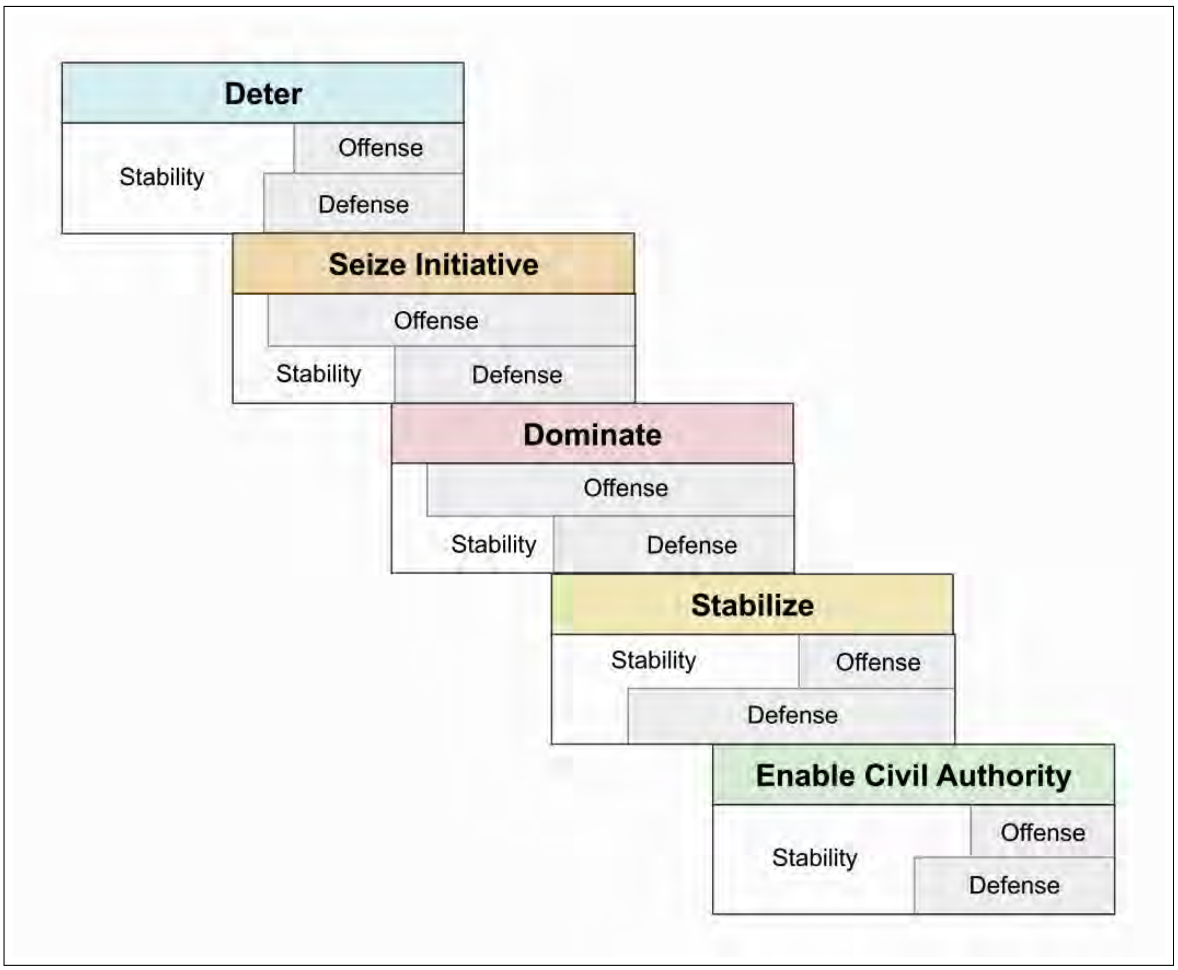


Figure 8. Notional Balance of Offense, Defense, and Stability Activities [22, p. V-15].

Other Planning Considerations.

Planners must ensure that their COAs pass the Feasible, Acceptable, Adequate, Distinguishable, and Complete (FAA-DC) test. Based on the facts and assumptions, strategies or COAs that are considered must actually be feasible in execution, they must be acceptable to the commander, they must be robust enough to complete the mission, they must differ from each other enough that they are easily distinguishable as separate options, and they must be fully planned out.

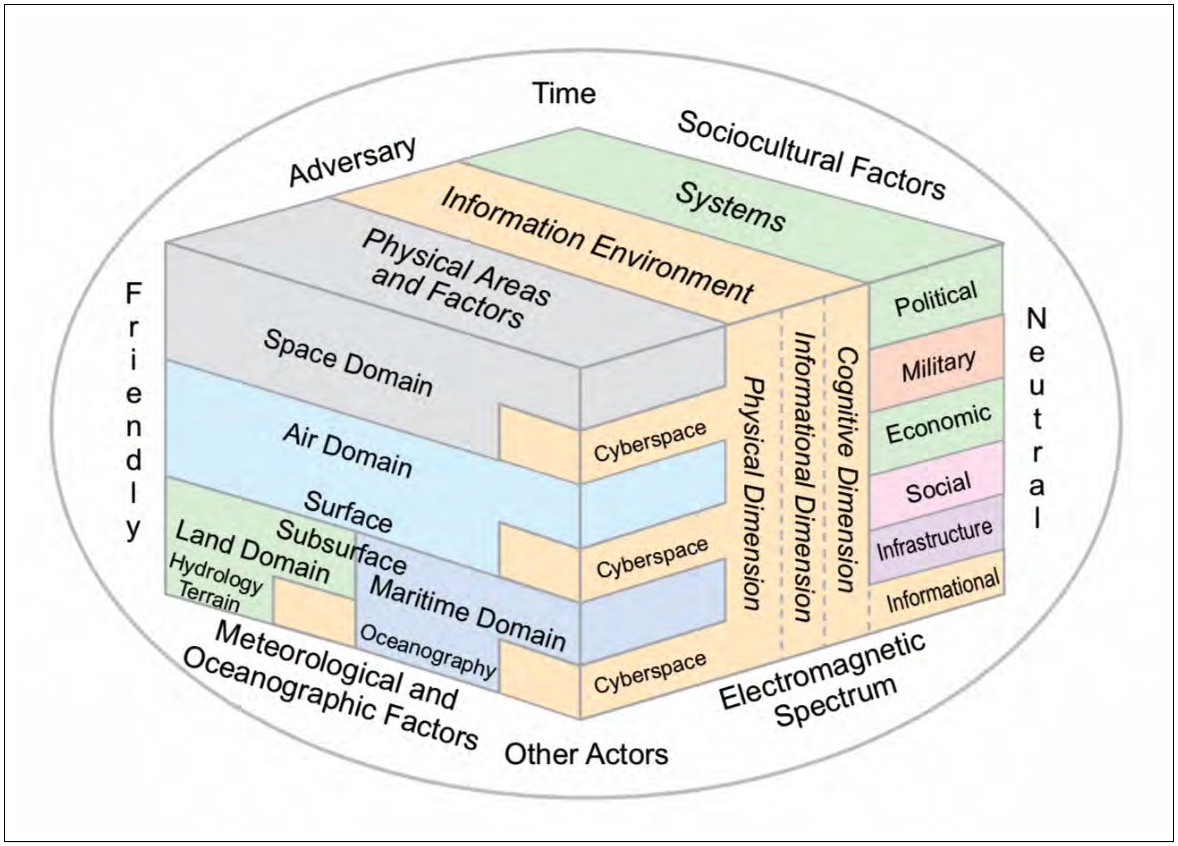


Figure 9. Holistic View of the Operational Environment [22, p. IV-12].

Tangredi wrote about the importance of considering a whole-of-government approach to countering antiaccess strategies [40]. Doctrinally, this is encapsulated in the Diplomatic, Information, Military, and Economic (DIME) activities. Additional considerations are the warfighting domains, Political, Military, Economic, Social, Infrastructure, and Information (PMESII) aspects of the operational environment, the adversaries, friendly actors, and time. Figure 9 shows the holistic view of the operational environment, including all of these factors and how they interact to form this complete picture. Effective planning will consider all of these aspects to create truly complete COAs for consideration.

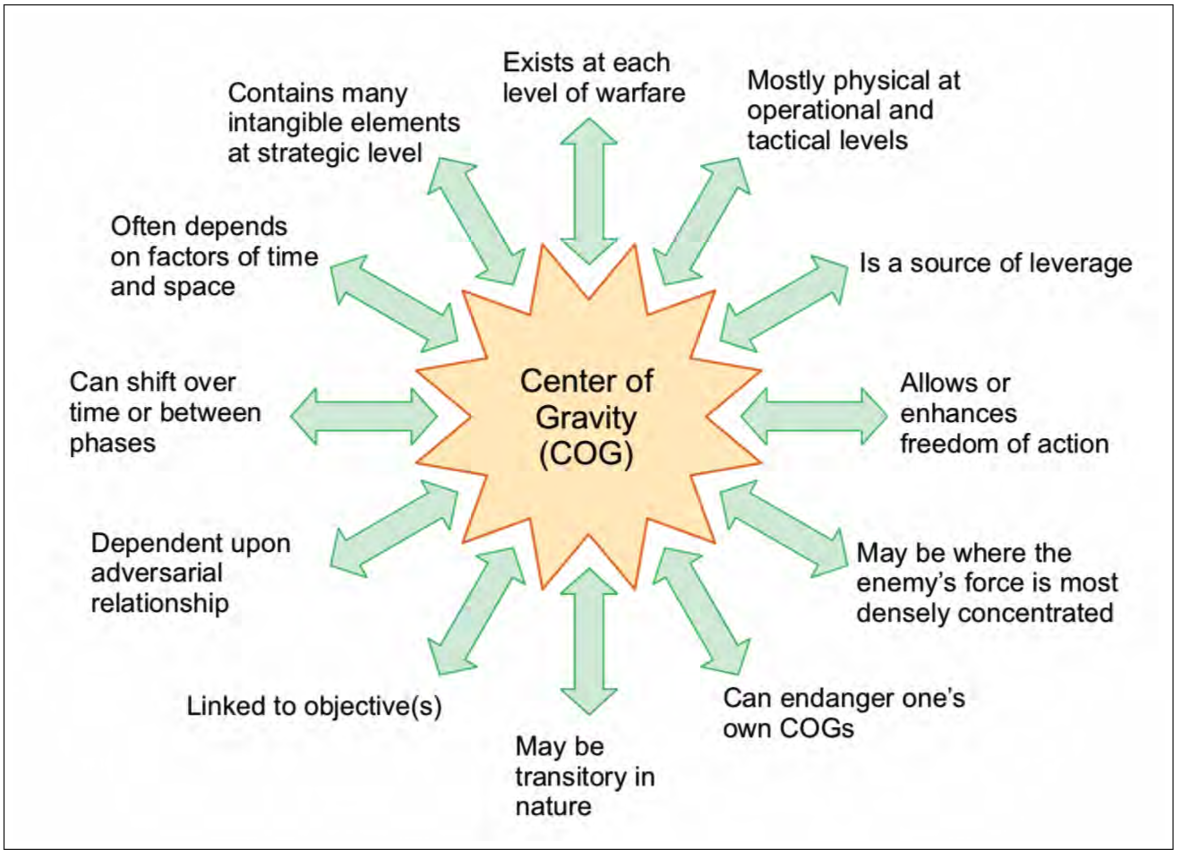


Figure 10. Characteristics of Centers of Gravity [22, p. IV-24].

Figure 10 shows how the Centers of Gravity (COGs) exist in an adversarial context and involve the clash of wills or physical strengths. COGs are defined by JP 5-0 [22, p. xxii] as, "a source of power that provides moral or physical strength, freedom of action, or will to act." As stated in the assumptions, this thesis will only deal with physical strengths as wills are often immeasurable. However, COGs must be analyzed for critical capabilities, critical requirements, and critical vulnerabilities. A proper analysis of adversary critical factors must be based on the best available knowledge of how adversaries organize, fight, think, and make decisions, and their physical and psychological strengths and weaknesses. Joint Force Commanders (JFCs) and their staffs must develop an understanding of their adversaries' capabilities and vulnerabilities, as well as factors that might influence an adversary to abandon its strategic objectives. They must also envision how friendly forces and actions appear from the adversaries' viewpoints. This will be critical to applying the method advanced in this thesis.

The Joint Intelligence Preparation of the Operational Environment.

Although Joint Publication 2-01.3, The Joint Intelligence Preparation of the Operational Environment, is not publicly available, JP 5-0 does have a great deal to say about the process. The JIPOE process is a comprehensive analytic tool to describe all aspects of the operational environment relevant to the operation or campaign [22, p. IV-10]. Situational awareness of the operational environment, especially including threats to national security, is the focus of JIPOE. This occurs during continuous monitoring of the national and international political and military situations. Commanders, planners, and staffs determine and analyze emerging crises, notify decision makers, and determine the specific nature of threats identified through ongoing JIPOE. The size and scope of the analysis depends on DIME and PMESII aspects of the operational environment [22, p. IV-10]. The JIPOE is continuously refined as planning requirements, considerations, and realities on the ground evolve and unfold. Planners can utilize the process to validate assumptions and ensure their COAs are FAA-DC.

Risk.

Central to planning and execution at any level is the concept of risk. Using the general strategy model of ends, ways, and means, risk results from the imbalance of these three components. JP 5-0 [22, p. B-3] defines risk as "probability and consequence of loss linked to hazards." This definition may not be entirely adequate for those advising senior leaders or conducting planning. The concept of risk resides firmly in the realm of decision-making. Risk has meaning when leaders weigh options to achieve desired objectives and assess the likelihood and magnitude of adverse outcomes. Those who write about risk often reside in academia or the business world where risks must be quantified to be useful. The discipline holds that risks can be accepted, avoided, transferred, or offset.

In most cases, military professionals first experience the concept of risk with the operational risk management process when risks are identified and controlled by educating subordinates and establishing measures to avoid or reduce the probability of negative outcomes. The two types of risk are Strategic Risk (risk to national interests) and Military Risk (risk to military objectives and to the Joint Force).

Another important source of guidance regarding risk is in the commander's intent for the campaign or operation. Purpose, end state, and operational risk are the essential elements of intent. An explicit statement of where, when, and what kinds of risk will be accepted or rejected provides a way to prioritize effort in the absence of resources and allows subordinate commanders to better execute mission command.

2.5 Complexity of the Problem

Wicked Problems.

In 1973, Rittel and Webber [36], both urban planners, observed that there is a whole realm of social planning problems that cannot be successfully treated with traditional linear, analytical approaches. They called these wicked problems, in contrast to straightforward, tame problems. Wicked problems are sets of complex, interacting issues evolving in a dynamic social context. New wicked problems often emerge as a result of trying to understand and solve the original wicked problem. Modeling warfare is a wicked problem because battles are not only a result of the objective characteristics of the opposing forces - they are ultimately battles of wills.

Although planning can provide an idea of what to expect, innumerable follow-on decisions and unknown effects can actually mold the eventual outcome. Planners can help commanders to think about scenarios before they happen and to identify possible outcomes, expected outcomes, or dangerous outcomes. In this way, planning prepares forces to deal with unexpected situations.

Proper contingency planning helps to break down a wicked problem into more manageable parts for further study. Every solution to a wicked problem is a "one-shot operation" because there is little opportunity to learn by trial-and-error; every attempt counts significantly.

Solution Complexity.

There are $\binom{n+m-1}{m-1}$ or $\left(\frac{(n+m-1)!}{(m-1)!n!}\right)$ ways to partition n resources across m fronts; as n and m become large, the problem quickly becomes computationally intractable [1]. As shown in Table 1 on page 20, even with only $n = 5$ resources fighting on $m = 3$ fronts, there are still $\binom{5+3-1}{3-1} = \binom{7}{2} = 21$ different strategies. Taking into account that is 21 strategies that each side has available, the total number of possible strategy profiles that should be evaluated would be $21 \times 21 = 441$.

Analyzing and keeping track of the large number of strategy profiles gets increasingly difficult as resources and fronts are increased. For example, consider evaluating an example where a percentage of total resources should be spread over an operating environment by each side by utilizing the six areas in the PMESII construct. For $n = 100$, one resource being a percentage point of total resource, and $m = 6$ for each of the six aspects of PMESII, the size of each player's strategy set becomes $\binom{100+6-1}{6-1} = \binom{105}{5} = 96,560,646$. Finally, once squared, the total possible strategy profiles becomes over 9.3 quadrillion.

Dimensionality and Feature Space.

A player's strategy set is the set of all strategies available to them to play a game. As models and games are approximations of reality, assumptions and simplifications must be made to translate a real-world problem into a mathematical model or game.

De Marchi [27] introduces the idea of a feature space and the assumptions that accompany it. Evaluating all of the dimensions on which various assumptions are made and exploring all of the sub-models within that space sounds good, but it quickly creates a problem - too many models to explore manually. To deal with the problem of too many models and too little time, de Marchi offeres a solution: use computational methods to explore the space of possible models. Beliefs are utilized to focus on sections of the feature space or strive to validate or further refine assumptions to try to narrow a strategy set based on its feasible characteristics. This approach requires combining art and science. One must constrain the feature space so that one is merely vexed and not cursed by the dimensionality.

The problem of having too many features describing an inductive learning task is the curse of dimensionality. As more features are added to the problem description, there are more features for the agent to use when constructing its hypothesis. More features make the model more expressive, but not all of these features may even be relevant to the concept. By relaxing such assumptions as winner-take-all payoffs or interactions between the fronts, i.e., resources in conflict on multiple fronts, a more realistic game can be created, but more complexity is introduced at the same time.

Fast computers may be able to analyze a large feature space, but someone (or something) must be able to build it and keep track of the results to effectively communicate them. Feature selection is used to remove many irrelevant features from the feature set and focus on what is important. Care taklen in creating a model will help control the feature space and enable a more productive analysis.

Probability and Uncertainty.

Probabilities or probability distributions provide the primary means for integrating uncertainty into a DSS. Sometimes, probabilities can be elicited by counting occurrences in sufficiently large data sets, or calculating an expected outcome given a random distribution of events. Sometimes, however, probabilities are best elicited from experts based on their opinion of what may happen. This is especially important in evaluating a one-shot occurrence, such as planning a campaign plan. Wars can be simulated to evaluate probable outcomes and strategy sets can be evaluated to find Nash Equilibria to determine the "best" strategy, but one-shot situations do not always follow these expected solutions.

Hora [21, p. 131] explains that a, "probability is a degree of belief and does not have a true, knowable value." As such, a probability elicited by an expert will vary according to the expert and their degree of belief. When creating a model, some aspect that could lead to a change in the calculated expected outcome might not be able to be efficiently included or included at all. In this case it may be valuable to consider expert opinion as a belief probability.

2.6 Summary

Success in the wars of the future will require conscious employment of interdependent operations across domains and time. The West recognizes this methodology in its likely adversaries, and those same adversaries have seen it in the action of the West. Military doctrine and campaign planning must evolve to successfully employ these operations in the future. The General Blotto game can be manipulated to account for various endogenous and exogenous parameters required to plan for future conflict.

Experiments are good for establishing trends and identifying tendencies. These tendencies are applicable as beliefs in one-shot games but not necessarily directly applicable to the application of this model.

Rational actors should pursue mixed strategy Nash equilibrium, but it is often hard to know if an actor is truly going to act rationally. Tendencies in strategies can be studied to understand the belief of acting as a rational actor [4]. Adversaries may not seem to act rationally or at least not act according to the other player's presumption of acting rationally. Therefore, mixed strategy calculations may not be as helpful as predicted. Although they are based on modeled assumptions, assumptions are uncertain until validated. Allowing for the substitution of beliefs to further this analysis until assumptions can be validated or the model can be otherwise improved is an important consideration in this thesis. Although it would be beneficial to "solve" for the most likely enemy COA, to remain most agile and adaptable to changing realities, responsible planners must consider a range of possible and dangerous enemy COAs to be most prepared to counter whatever strategy materializes.

In 1981, Shubik and Weber [38] successfully applied the Blotto model to military and systems defense by work. In 1983, Grotte and Brooks [17] used Blotto games to measure naval presence. Some additional works of note that were considered while evaluating this problem and formulating a framework include: Bullock, Deckro, and Weir's [7] combination of Value Focused Thinking (VFT) with game theory to produce a mixed strategy for each player based on their respective value hierarchy, and Collins and Hester's [10] combination of a Blotto game with Lanchester equations. Although these offer promise for future research, this thesis focuses on the large feature space created by a complex Blotto game.

III. Methodology

In forming the plan of a campaign, it is requisite to foresee everything the enemy may do, and to be prepared with the necessary means to counteract it. Plans of campaign may be modified, ad infinitum, according to circumstances – the genius of the general, the character of the troops, and the topography of the theater of action.

- Napoléon Bonaparte[1]

3.1 Overview

This chapter discusses the method of construction of the General Blotto game and its application to campaign planning, especially to situations with complex, uncertain, non-linear, endogenous and exogenous factors in multiple domains and across multiple phases. This thesis is motivated by Tangredi [40], who argues for the need to consider multiple domains and all available resources in the various phases of campaign execution. The scope of this thesis is to design and construct a suitable proof of concept through schema formulation, data-driven parameters, methods of calculation, scenarios, and other considerations. Given an appropriate deconstruction of the elements of operational design, this model is applied to the Joint Campaign Planning Methodology (see Figure 5) to create a simple and useful method for framing the development of COAs. In this chapter, a mathematical construction of the General Blotto framework is created to apply to the example problem.

Much as planners work with the commander to develop the commander's evaluation criteria early in the COA development process, a similar process can frame criteria for COA development. Ideally, multiple frameworks can be utilized to help in the COA development process. This framework defines the resources and fronts of the

[1]Maxim II, quoted from p. 10 of "The Officer's Manual. Napoleon's Maxims of War," translated from French by Col. D'Aguilar, and published by West & Johnston of Richmond, VA, in 1862. Napoléon thought too much in war could be ascribed to luck or unforeseen circumstances. Generals cannot count on events happening, they must take advantage of opportunities as they see them.

General Blotto game. Some examples include domains and theater resources, secured forward medical facilities and airfields and evacuation and recovery assets, or secure forward munitions depots and limited-supply strategic munitions. A framework that establishes various tasks as fronts could be setup, with mission essential tasks given higher weights and less important tasks given lower weights.

The JIPOE informs the adversary's capability relative to each measure and the probable importance or value they may give to each front. The relative weights that each side gives to a front, whether it be a domain, terrain, event, or task need not be the same, as one side may see more strategic importance in one than the other.

The goal is to gain insight into the best potential strategies to counter the most probable (or dangerous) strategies of an adversary. Because this thesis is concerned with narrowing feature spaces in which to develop COAs, not creating an unwieldy model to analyze, it is best to carefully choose the fronts and resources. They must be applicable to framing possible COAs that are FAA-DC. First, carefully choose and define the adversaries, or players. Next, define the fronts and assign resources. Then, determine the victors and score the payoffs. The payoffs correspond to more or less successful victories and defeats. By controlling for and changing how the game is scored, planners can easily explore nuances in the model and their plans. Once the payoff matrix is created, mixed strategies can be explored, showing how the game might play out between random opponents. Because warfare is a one-off game, and the likelihoods and tendencies of the players will have been included in the JIPOE, these computed mixed strategies may be replaced with belief probabilities that help to focus the problem and narrow the feature space. This allows the planners to focus on developing successful strategies.

3.2 The General Blotto Framework

Asymmetric Two-Player Game.

Generalizing the Colonel Blotto game allows for an asymmetric, two-player game. A game where each player's strategy sets differ from the other's. Initially, this will most likely be due to asymmetrically sized resource pools when one side has more or less resources than another. It may also occur with the same size resource pools, but differing constraints or limitations on the ways those resources can be distributed will lead to different, asymmetric strategy sets. Like the original, General Blotto is still a use-it-or-lose-it game, so the asymmetry allows for only the resources being used by either side to be defined as the resources available for the purposes of the game. Because this method models conflict, it is assumed that one side is on the offensive and one side is on the defensive. Following this assumption, any draws will benefit the defender as a stalemate means that the defenses "have not been overrun."

In introducing the Colonel Blotto game, the same notation as Golman and Page [14] is utilized. In generalizing the game, this thesis transitions to the game theory notation used by Watson [43, p. 23-27]. To model the two-player game, the first player, the defensive player, is called Player 1. The second player, the offensive player, is called Player 2. In allowing for asymmetric forces, the number of resources, n, must be distinguishable by player. As such, n_i is introduced, where $i \in \{1, 2\}$, as the number of resources for Player 1 and Player 2, respectively.

On Heterogeneous Fronts.

The game is played across m heterogeneous fronts, each of which are distinguishable, allowing for fronts to matter as individual, specific places. Each player values each front according to their own criteria. The players have a strategy set, S, which contains all of the possible strategy profiles. S consists of Player 1's strategy set, S_1, and Player 2's strategy set, S_2. The size of each S_i is dependent on they number of ways to partition n_i resources on m fronts, so there exists a maximum of $\frac{(n_i+m-1)!}{(m-1)!n_i!}$ distinguishable strategies available to each player. For all games in this thesis, assume $m, n, n_i \in \mathbb{N}$ and $m, n, n_i \geq 2$. For Player 1 there are j possible strategies in S_1, where:

$$j = \frac{(n_1 + m - 1)!}{(m-1)!n_1!}, \tag{4}$$

and for Player 2 there are k possible strategies in S_2, where:

$$k = \frac{(n_2 + m - 1)!}{(m-1)!n_2!}. \tag{5}$$

A strategy profile, s, is a vector of strategies, s_i, for each player in the form: $s = (s_1, s_2), s \in S$. Player 1's strategy, s_1, is written as a real vector of x_i budget allocations:

$$s_1 = (x_1, x_2, \ldots, x_m), s_1 \in S_1, \tag{6}$$

and Player 2's strategy, s_2, is similarly written as a real vector of y_i budget allocations:

$$s_2 = (y_1, y_2, \ldots, y_m), s_2 \in S_2. \tag{7}$$

Recall equation (1) from Golman and Page [14]:

$$\sum_{i=1}^{m} x_i^{CB} = 1, x_i^{CB} \in [0, 1],$$

where x_i^{CB} is in the continuous interval from zero to one and represents the fraction of the budget allocation to front i, and m is the total number of fronts. One adjusts this equation for Player 1 and the similar equation (2) for Player 2 to bring it in line with Watson's notation [43, p. 23-27] and to more clearly denote the strategies as non-negative integer portions, x_i and y_i, of the entire number of resources, n_i, that each player has to distribute. For Player 1, this change is written:

$$\sum_{i=1}^{m} x_i = n_1, x_i \in \mathbb{N}_{\geq 0}, \tag{8}$$

and for Player 2:

$$\sum_{i=1}^{m} y_i = n_2, y_i \in \mathbb{N}_{\geq 0}. \tag{9}$$

Finally, one assigns utility vectors according to how much, w_i^m, each player i weighs the importance of each front, 1 through m. The payoff function, $u_i(\mathrm{s})$, is the utility realized by player i upon winning each strategy profile, s. The weights are the real vectors:

$$(w_1^1, w_1^2, \ldots, w_1^m), 0 \leq u_1 \leq \sum_{i=1}^{m} w_1^i \tag{10}$$

and

$$(w_2^1, w_2^2, \ldots, w_2^m), 0 \leq u_2 \leq \sum_{i=1}^{m} w_2^i. \tag{11}$$

Determining Victory.

To determine who wins in a matchup of each strategy, the first step is to score each of the m fronts according to how many resources are assigned to each. In this first step, a win is worth one point, a loss is worth minus one point, and a draw awards one point to the defensive player, Player 1. The second step is to sum the scores of the fronts to determine the winner of the war between these two strategies. The equation for these two steps to determine the victor between Player 1 and Player 2 is:

$$\sum_{i=1}^{m} \mathrm{sgn}(x_i - y_i), \text{ where } \mathrm{sgn}(\chi) := \begin{cases} 1 & \text{if } \chi \geq 0, \\ -1 & \text{if } \chi < 0. \end{cases} \tag{12}$$

Positive payoffs for Player 1 and negative payoffs for Player 2 are canonical for two-player games. Therefore, if equation (12) is positive, Player 1 has the winning strategy, while Player 2 wins for a negative sum. If the result is zero (a draw) then assume the defender wins.

Note that similar to the alternative payoff functions proposed by Golman and Page [14, p. 286], any appropriate method may be alternatively used to determine the winning strategies. Situations where more or less than a majority of fronts, or requiring combinations of fronts to be won could be feasible. This thesis will only use equation (12) to determine winning strategies.

Scoring the Payoff Matrix.

Once the winner of each strategy profile is determined, $s = (s_1, s_2) \forall s \in S$, calculate each payoff according to the payoff function, $u_i(s)$. To do this, populate a payoff matrix where the player's payoff functions, $u_i(s)$ are defined over S. This is a $j \times k$ matrix, where each entry is the calculated payoff, or utility to the winning player. Player 1 is the winner for a utility with a non-negative sign and Player 2 is the winner for a utility with a negative-sign. Given the weight vectors, (10) and (11), the player who wins a front (outright) earns their weight given to that front while a draw or loss earns no points. Therefore, where Player 1 has the winning strategy, the payoff would be:

$$u_{jk} = \sum_{i=1}^{m} \operatorname{sgn}(x_i - y_i) w_1^i, \text{ where } \operatorname{sgn}(\phi) := \begin{cases} 1 & \text{if } \phi > 0, \\ 0 & \text{if } \phi = 0, \\ -1 & \text{if } \phi < 0, \end{cases} \tag{13}$$

and where Player 2 has the winning strategy, the payoff would be:

$$u_{jk} = \sum_{i=1}^{m} \text{sgn}(x_i - y_i) w_2^i, \text{ where } \text{sgn}(\phi) := \begin{cases} 1 & \text{if } \phi > 0, \\ 0 & \text{if } \phi = 0, \\ -1 & \text{if } \phi < 0. \end{cases} \tag{14}$$

Table 10 shows an example payoff matrix, U, for General Blotto game.

Table 10. Example normal form General Blotto game.

	Attacker			
Defender	s_2^1	s_2^2	$\cdots$	s_2^k
s_1^1	u_{11}	u_{12}	$\cdots$	u_{1k}
s_1^2	u_{21}	u_{22}	$\cdots$	u_{2k}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s_1^j	u_{j1}	u_{j2}	$\cdots$	u_{jk}

Just as it would be feasible to alter equation (12), one can alter the payoff equations, (13) and (14), so that certain combinations of fronts allow for larger or differing payoffs, i.e., winning Front 1 and Front 2 is better than winning Front 2 and Front 3 and both are better than winning Front 1 and Front 3. This thesis only utilizes equations (13) and (14).

Using Microsoft Excel to calculate payoffs and display color-coded matrices is useful as a visual aide in analyzing the feasible feature space. Algorithm 1 shows how to include equations (12), (13), and (14) to determine winners and payoffs.

Calculating the Mixed Strategies.

A Nash equilibrium is given by a mixed strategy for each player that is a best response to the fixed strategy of the other player. σ is the mixed strategy profile of the two players, $\sigma = (\sigma_1, \sigma_2)$, where $\sigma_1 = (p_1^1, p_1^2, \ldots, p_1^j)$ is a mixed strategy available to Player 1 and $\sigma_2 = (p_2^1, p_2^2, \ldots, p_2^k)$ is a mixed strategy available to Player 2. σ_1

Algorithm 1 Calculate the Payoff Matrix

```
function UTILITY(S)                      ▷ u_i(s) is the payoff function for player i
    for all s ∈ S do                     ▷ Compare all strategy profiles, s = (s_1, s_2)
        sum ← 0                          ▷ Initialize a summation variable ∀s ∈ S
        for i ← 1, m do                  ▷ Evaluate the matchup on every front
            if x_i ≥ y_i then            ▷ Use eq. (12)
                sum ← sum + 1            ▷ +1 for Player 1 win
            else
                sum ← sum − 1            ▷ −1 for Player 2 win
            end if
        end for
        if sum ≥ 0 then                  ▷ If Player 1 won the strategy profile, use eq. (13)
            sum ← 0
            for i ← 1, m do              ▷ Evaluate utility awards on every front
                if x_i > y_i then        ▷ If Player 1 won the front...
                    sum ← sum + w_1^i    ▷ Add front weight to Player 1 utility
                else
                    sum ← sum + 0        ▷ No weight added to Player 1 utility
                end if
                u ← sum                  ▷ Assign Player 1 utilities where earned ∀u ∈ U
            end for
        else                             ▷ If Player 2 won the strategy profile, use eq. (14)
            sum ← 0
            for i ← 1, m do              ▷ Evaluate utility awards on every front
                if y_i > x_i then        ▷ If Player 2 won the front...
                    sum ← sum − w_2^i    ▷ Add (−) front weight to Player 2 utility
                else
                    sum ← sum − 0        ▷ No weight added to Player 2 utility
                end if
                u ← sum                  ▷ Assign Player 2 utilities where earned ∀u ∈ U
            end for
        end if
    end for                              ▷ Utilities have been assigned ∀s ∈ S
end function
```

is a probability distribution over the j pure strategies in S_1 and σ_2 is a probability distribution over the k pure strategies in S_2. The probabilities, p_i, for each pure strategy, s_i are the limits of their relative frequency in a large number of trials. If the players played their pure strategies randomly, over multiple iterations, p_1, where $p_1 \in \mathbb{R}, 0 \leq p_1 \leq 1$, and $\sum_j p_1^j = 1$, is the probability that Player 1 plays a pure strategy, s_1. Likewise, p_2, where $p_2 \in \mathbb{R}, 0 \leq p_2 \leq 1$, and $\sum_k p_2^k = 1$, is the probability that Player 2 plays a pure strategy, s_2.

Table 11. Example normal form General Blotto game with mixed strategies.

		Attacker			
		s_2^1	s_2^2	$\cdots$	s_2^k
Defender	Mixed Strategy	p_2^1	p_2^2	$\cdots$	p_2^k
s_1^1	p_1^1	u_{11}	u_{12}	$\cdots$	u_{1k}
s_1^2	p_1^2	u_{21}	u_{22}	$\cdots$	u_{2k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s_1^j	p_1^j	u_{j1}	u_{j2}	$\cdots$	u_{jk}

Table 11 shows an example normal form General Blotto game with mixed strategies, $\sigma_1 = (p_1^1, p_1^2, \ldots, p_1^j)$ and $\sigma_2 = (p_2^1, p_2^2, \ldots, p_2^k)$, calculated for every pure strategy profile in each player's strategy sets, S_1 and S_2. Calculating and showing these mixed strategies is feasible for games with small feature spaces, but quickly becomes difficult as the number of pure strategies increases into the many tens and hundreds.

Analyzing the Strategies.

Calculating the mixed strategies for a game with a small feature space gives an idea of how a particular strategy would do over multiple iterations of random play by each player, or an expected outcome. However, planners also conduct JIPOE to inform beliefs of the other player's most probable strategies, or what is believed the probabilities are that they might play each remaining strategy. In reality, unlike a game, real players are constrained by movement, readiness, budgets, or any host of variables, many of which cannot readily be modeled or are completely unforeseen. The better one can model or analyze or predict these actual capabilities, the more realistic one can make this game and the more likely one is to find a strategic advantage. Conversely, the more complex the model is, the more difficult it is to solve or analyze.

To determine how a conflict might look in reality, rather than the likelihood of an outcome in a random sampling, especially in a very large feature space, expert beliefs that the adversary will play a particular pure strategy, θ_i, may be substituted. θ_1 is the belief that Player 1 might play a particular strategy, s_1, while θ_2 is the belief that Player 2 might play a particular strategy, s_2. Table 12 shows an example payoff matrix for General Blotto game with these expert-elicited beliefs.

Table 12. Example normal form General Blotto game with Bayesian probabilities.

		Attacker			
		s_2^1	s_2^2	$\cdots$	s_2^k
Defender	Beliefs	θ_2^1	θ_2^2	$\cdots$	θ_2^k
s_1^1	θ_1^1	u_{11}	u_{12}	$\cdots$	u_{1k}
s_1^2	θ_1^2	u_{21}	u_{22}	$\cdots$	u_{2k}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
s_1^j	θ_1^j	u_{j1}	u_{j2}	$\cdots$	u_{jk}

Any strategies that the adversary either will not play because they are dominated, or it is believed that they are very highly unlikely to play can be removed. Which strategies are filtered out is a judgement made depending on the size of the space and the resources available to the planners and commander to investigate the COAs.

3.3 Conclusion

The methodology outlined in this chapter achieves the objectives proposed in Chapter I by considering and incorporating the research compiled in Chapter II. This chapter provides an explanation of how the General Blotto game is utilized to provide a framework for campaign planning and analysis. Chapter IV discusses the analysis of illustrative instances and a scenario involving this method and its results.

IV. Analysis and Results

The tactical result of an engagement forms the base for new strategic decisions because victory or defeat in a battle changes the situation to such a degree that no human acumen is able to see beyond the first battle... Therefore no plan of operations extends with any certainty beyond the first contact with the main hostile force.

- Helmuth von Moltke the Elder[1]

4.1 Overview

This chapter presents, sets up, and discusses illustrative instances of the General Blotto game. By working through simple examples, it illustrates how the model may be utilized in real-world situations. An example situation is also given, with example assumptions and simple but realistic parameters. This is used as a proof of concept for the use of the General Blotto game approach for planners and commanders to understand the feature spaces of complex planning problems and how this model may help them to gain insight. The results are discussed and analyzed.

4.2 General Blotto Illustrative Instances

To demonstrate and explain the methodology and how it is applied, a number of illustrative instances are presented. These instances are designed to illustrate the concepts discussed in the methodology and how the General Blotto game is useful for planners and commanders to gain insight into their campaign plans.

[1]Field Marshal credited with creating modern methods for directing armies in the field. Helmuth von Moltke is quoted from his own 1871 book, *On Strategy* as translated in: Hughes, Daniel J. and Harry Bell. *Moltke on the Art of War: Selected Writings.* Novato, CA: Presidio Press, 1993:92.

Symmetric vs. Asymmetric Forces.

Symmetric games are fair because both players have equal resources, so it is assumed that their skills will be the deciding factor in victory. War is not a game. Ideally, it should not be fair. When lives are on the line, one wants one's own side to win and to win decisively, with as few casualties as possible. This means that if one must go to war, bring the largest, strongest, most capable force one can. Figure 11 shows a simple, normal form Colonel Blotto game between two players. They each have $n = 4$ resources and are fighting across $m = 2$ fronts, for simplicity's sake. Because draws are won by defenders in this model, Player 2 must change strategies

		Attacker				
		(4,0)	(0,4)	(3,1)	(1,3)	(2,2)
Defender		0.200	0.200	0.200	0.200	0.200
(4,0)	0.200	0	1	1	1	1
(0,4)	0.200	1	0	1	1	1
(3,1)	0.200	1	1	0	1	1
(1,3)	0.200	1	1	1	0	1
(2,2)	0.200	1	1	1	1	0

Figure 11. Illustrative instance of the General Blotto game with symmetric forces ($S_1 \equiv S_2$) and calculated mixed strategies. Strategies colored green are probabilistically better than those labeled red.

to win. In Figure 12, Player 2 has brought more resources to the fight, with $n_2 = 6$. In this asymmetric battle, Player 2, the attacker, has a winning strategy for every

		Attacker						
		(6,0)	(0,6)	(5,1)	(1,5)	(4,2)	(2,4)	(3,3)
Defender		0.000	0.000	0.250	0.250	0.150	0.150	0.200
(4,0)	0.250	0	1	-2	1	0	1	1
(0,4)	0.250	1	0	1	-2	1	0	1
(3,1)	0.150	1	1	0	1	-2	1	0
(1,3)	0.150	1	1	1	0	1	-2	0
(2,2)	0.200	1	1	1	1	0	0	-2

Figure 12. Illustrative instance of the General Blotto game with asymmetric forces ($S_1 \neq S_2$) and calculated mixed strategies. Dark blue and dark red correspond to more decisive wins by the Defender and Attacker, respectively. Strategies colored green are probabilistically better than those labeled red.

one of Player 1's defensive strategies.

Increasing the Fronts.

In an asymmetric fight, the side with the inferior, smaller, weaker, less capable, or less advanced forces must avoid simple head-to-head conflict with the stronger adversary. The General Blotto game, along with historic examples, show that by increasing the number of fronts, the weaker side can extend the stronger side's forces, making individual battles more able to be won. For example, terrorists tend to target lone symbols of their foe's status or power instead of taking on their military on an open battlefield. The fronts become all symbols of that country's influence around the world. The country must decide how it will protect all of its interests. The terrorists must only find the weakest target and may attack that. The successful attack may not win the war, but causes the country to re-allocate its resources, again potentially leaving other targets open. In this example, the terrorists have effectively used a multiple-shot General Blotto methodology to fight a stronger adversary by increasing the number of fronts in play. Figure 13 shows an instance of a stronger defender,

		Attacker				
		(4,0)	(0,4)	(3,1)	(1,3)	(2,2)
Defender		0.200	0.200	0.200	0.200	0.200
(5,0)	0.167	1	1	1	1	1
(0,5)	0.167	1	1	1	1	1
(4,1)	0.167	1	1	1	1	1
(1,4)	0.167	1	1	1	1	1
(3,2)	0.167	1	1	1	1	1
(2,3)	0.167	1	1	1	1	1

Figure 13. Illustrative instance of being dominated by a stronger adversary on only a few fronts. Calculated mixed strategies show the ambivalence of the method of domination.

Player 1 with $n_1 = 5$, completely dominating a weaker attacker, Player 2 with $n_2 = 4$ resources, on $m = 2$ fronts. The attacker discovers another attack vector, and moves to reposition their forces for an attack on $m = 2$ fronts. Figure 14 shows how, with the same number of forces, Player 2 may have a chance in successfully attacking Player 1, depending on the strategies chosen (Player 1 can still dominate if they know to

		Attacker														
		(4,0,0)	(0,4,0)	(0,0,4)	(3,1,0)	(3,0,1)	(1,3,0)	(1,0,3)	(0,3,1)	(0,1,3)	(2,2,0)	(2,0,2)	(0,2,2)	(2,1,1)	(1,2,1)	(1,1,2)
Defender		0.000	0.000	0.000	0.032	0.032	0.048	0.048	0.063	0.063	0.269	0.269	0.176	0.000	0.000	0.000
(5,0,0)	0.000	1	1	1	1	1	1	1	-2	-2	1	1	-2	-2	-2	-2
(0,5,0)	0.000	1	1	1	1	-2	1	-2	1	1	1	-2	1	-2	-2	-2
(0,0,5)	0.000	1	1	1	-2	1	-2	1	1	1	-2	1	1	-2	-2	-2
(4,1,0)	0.000	1	1	2	1	2	1	2	-2	1	1	2	-2	1	-2	1
(4,0,1)	0.000	1	2	1	2	1	2	1	1	-2	2	1	-2	1	1	-2
(1,4,0)	0.000	1	1	2	1	-2	1	1	2	2	1	-2	2	-2	1	1
(1,0,4)	0.000	1	2	1	-2	1	1	1	2	2	-2	1	2	-2	1	1
(0,4,1)	0.000	2	1	1	2	1	2	-2	1	1	2	-2	1	1	1	-2
(0,1,4)	0.000	2	1	1	1	2	-2	2	1	1	-2	2	1	1	-2	1
(3,2,0)	0.167	1	1	2	1	1	1	2	-2	2	1	2	1	2	1	2
(3,0,2)	0.167	1	2	1	1	1	2	1	2	-2	2	1	1	2	2	1
(2,3,0)	0.095	1	1	2	1	-2	1	2	1	2	1	1	2	1	2	2
(2,0,3)	0.095	1	2	1	-2	1	2	1	2	1	1	1	2	1	2	2
(0,3,2)	0.024	2	1	1	2	2	1	-2	1	1	2	1	1	2	2	1
(0,2,3)	0.024	2	1	1	2	2	-2	1	1	1	1	2	1	2	1	2
(3,1,1)	0.000	2	2	2	1	1	2	2	1	1	2	2	-2	1	1	1
(1,3,1)	0.000	2	2	2	2	1	1	1	1	2	2	-2	2	1	1	1
(1,1,3)	0.000	2	2	2	1	2	1	1	2	1	-2	2	2	1	1	1
(2,2,1)	0.000	2	2	2	2	1	2	2	1	2	1	1	1	1	1	2
(2,1,2)	0.000	2	2	2	1	2	2	2	2	1	1	1	1	1	2	1
(1,2,2)	0.429	2	2	2	2	2	1	1	2	2	1	1	1	2	1	1

Figure 14. Illustrative instance of increasing the fronts to diminish the advantage of a stronger adversary with calculated mixed strategies. Dark blue and dark red correspond to more decisive wins by the Defender and Attacker, respectively.

play the strategy $s_2 \in \{(2,2,1),(2,1,2),(1,2,2)\}$).

Homogeneous vs. Heterogeneous Fronts.

Although making one front more important that other fronts immediately makes the game more realistic, it does not in itself change the outcome of the game using the methodology set forth in this initial effort. Equations (13) and (14) illustrate that payoffs are calculated according to the methods most often employed in the original Colonel Blotto game [14]. More complex methods of payoff calculation could allow for circumstances where groups of fronts captured together yield higher payoffs that those fronts otherwise pay off separately. If the payoffs are the same between players, winning fewer but more valuable fronts could be more efficient than winning a majority of fronts. An example of this would be winning the Electoral College in a United States Presidential election while perhaps losing the national popular vote.

		Attacker																				
		(5,0,0)	(0,5,0)	(0,0,5)	(4,1,0)	(4,0,1)	(1,4,0)	(1,0,4)	(0,4,1)	(0,1,4)	(3,2,0)	(3,0,2)	(2,3,0)	(2,0,3)	(0,3,2)	(0,2,3)	(3,1,1)	(1,3,1)	(1,1,3)	(2,2,1)	(2,1,2)	(1,2,2)
Defender		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.111	0.111	0.111	0.111	0.111	0.111	0.111	0.111	0.111	0.000	0.000	0.000
(5,0,0)	0.000	0	1	1	1	1	1	1	-2	-2	1	1	1	1	-2	-2	-2	-2	-2	-2	-2	-2
(0,5,0)	0.000	1	0	1	1	-2	1	-2	1	1	1	-2	1	-2	1	1	-2	-2	-2	-2	-2	-2
(0,0,5)	0.000	1	1	0	-2	1	-2	1	1	1	-2	1	-2	1	1	1	-2	-2	-2	-2	-2	-2
(4,1,0)	0.000	1	1	2	0	1	1	2	-2	1	1	2	1	2	-2	-2	1	-2	1	-2	1	-2
(4,0,1)	0.000	1	2	1	1	0	2	1	1	-2	2	1	2	1	-2	-2	1	1	-2	1	-2	-2
(1,4,0)	0.000	1	1	2	1	-2	0	1	1	2	1	-2	1	-2	2	2	-2	1	1	-2	-2	1
(1,0,4)	0.000	1	2	1	-2	1	1	0	2	1	-2	1	-2	1	2	2	-2	1	1	-2	-2	1
(0,4,1)	0.000	2	1	1	2	1	1	-2	0	1	2	-2	2	-2	1	1	1	1	-2	1	-2	-2
(0,1,4)	0.000	2	1	1	1	2	-2	1	1	0	-2	2	-2	2	1	1	1	-2	1	-2	1	-2
(3,2,0)	0.111	1	1	2	1	-2	1	2	-2	2	0	1	1	2	-2	1	1	-2	2	1	2	1
(3,0,2)	0.111	1	2	1	-2	1	2	1	2	-2	1	0	2	1	1	-2	1	2	-2	2	1	1
(2,3,0)	0.111	1	1	2	1	-2	1	2	-2	2	1	-2	0	1	1	2	-2	1	2	1	1	2
(2,0,3)	0.111	1	2	1	-2	1	2	1	2	-2	-2	1	1	0	2	1	-2	2	1	1	1	2
(0,3,2)	0.111	2	1	1	2	2	-2	-2	1	1	2	1	1	-2	0	1	2	1	-2	2	1	1
(0,2,3)	0.111	2	1	1	2	2	-2	-2	1	1	1	2	-2	1	1	0	2	-2	1	1	2	1
(3,1,1)	0.111	2	2	2	1	1	2	2	1	1	1	1	2	2	-2	-2	0	1	1	1	1	-2
(1,3,1)	0.111	2	2	2	2	1	1	1	1	2	2	-2	1	-2	1	2	1	0	1	1	-2	1
(1,1,3)	0.111	2	2	2	1	2	1	1	2	1	-2	2	-2	1	2	1	1	1	0	-2	1	1
(2,2,1)	0.000	2	2	2	2	1	2	2	1	2	1	-2	1	1	-2	1	1	1	2	0	1	1
(2,1,2)	0.000	2	2	2	1	2	2	2	2	1	-2	1	1	1	1	-2	1	2	1	1	0	1
(1,2,2)	0.000	2	2	2	2	2	1	1	2	2	1	1	-2	-2	1	1	2	1	1	1	1	0

Figure 15. Illustrative instance of the General Blotto game with homogeneous fronts ($w_i = w_{i+1}$) and calculated mixed strategies.

This thesis allows for the weighting of fronts to be able to explore the relationship between otherwise similar outcomes when one front is favored over others. In Figure 14, Player 2 has some chance across most strategies, but is still dominated by three of Player 1's strategies. Figure 15 shows that Player 2 has learned to use more force to have a chance at a successful attack under all of Player 1's strategy profiles, but it is hard to immediately determine an underlying preference, as the payoff for Player 2 winning is -2 in all cases. Figure 16 shows the Microsoft Excel setup changing the weights applied to each front. Now, the players have been given preferences for their fronts - the fronts are distinguishable. Figure 17 shows how the payoffs have changed according to the player preferences for particular fronts. A more likely enemy COA or more favorable friendly COA may be gleaned from such additional information.

Narrowing the Feature Space.

In the case that the game is symmetric, Hart [19] found that for n discrete indivisible units to be allocated between m fronts, the mixed strategies were on $[0, \frac{2n}{m}]$. This will still hold for this formulation, because the calculated utilities do not allow for a

	Defender Strategies		Armies:	5			
			Fronts:	3	Front 1	Front 2	Front 3
21	total strategy permutations				weight:	weight:	weight:
#	Strategy	Permutation			3	1	2
1	01	A	DS01A	(5,0,0)	5	0	0
2	01	B	DS01B	(0,5,0)	0	5	0
3	01	C	DS01C	(0,0,5)	0	0	5
4	02	A	DS02A	(4,1,0)	4	1	0
5	02	B	DS02B	(4,0,1)	4	0	1
6	02	C	DS02C	(1,4,0)	1	4	0
7	02	D	DS02D	(1,0,4)	1	0	4
8	02	E	DS02E	(0,4,1)	0	4	1
9	02	F	DS02F	(0,1,4)	0	1	4
10	03	A	DS03A	(3,2,0)	3	2	0
11	03	B	DS03B	(3,0,2)	3	0	2
12	03	C	DS03C	(2,3,0)	2	3	0
13	03	D	DS03D	(2,0,3)	2	0	3
14	03	E	DS03E	(0,3,2)	0	3	2
15	03	F	DS03F	(0,2,3)	0	2	3
16	04	A	DS04A	(3,1,1)	3	1	1
17	04	B	DS04B	(1,3,1)	1	3	1
18	04	C	DS04C	(1,1,3)	1	1	3
19	05	A	DS05A	(2,2,1)	2	2	1
20	05	B	DS05B	(2,1,2)	2	1	2
21	05	C	DS05C	(1,2,2)	1	2	2

	Attacker Strategies		Armies:	5			
			Fronts:	3	Front 1	Front 2	Front 3
21	total strategy permutations				weight:	weight:	weight:
#	Strategy	Permutation			1	2	3
1	01	A	AS01A	(5,0,0)	5	0	0
2	01	B	AS01B	(0,5,0)	0	5	0
3	01	C	AS01C	(0,0,5)	0	0	5
4	02	A	AS02A	(4,1,0)	4	1	0
5	02	B	AS02B	(4,0,1)	4	0	1
6	02	C	AS02C	(1,4,0)	1	4	0
7	02	D	AS02D	(1,0,4)	1	0	4
8	02	E	AS02E	(0,4,1)	0	4	1
9	02	F	AS02F	(0,1,4)	0	1	4
10	03	A	AS03A	(3,2,0)	3	2	0
11	03	B	AS03B	(3,0,2)	3	0	2
12	03	C	AS03C	(2,3,0)	2	3	0
13	03	D	AS03D	(2,0,3)	2	0	3
14	03	E	AS03E	(0,3,2)	0	3	2
15	03	F	AS03F	(0,2,3)	0	2	3
16	04	A	AS04A	(3,1,1)	3	1	1
17	04	B	AS04B	(1,3,1)	1	3	1
18	04	C	AS04C	(1,1,3)	1	1	3
19	05	A	AS05A	(2,2,1)	2	2	1
20	05	B	AS05B	(2,1,2)	2	1	2
21	05	C	AS05C	(1,2,2)	1	2	2

Figure 16. Illustrative instance showing the Microsoft Excel setup changing the weights applied to each front.

		Attacker																				
		(5,0,0)	(0,5,0)	(0,0,5)	(4,1,0)	(4,0,1)	(1,4,0)	(1,0,4)	(0,4,1)	(0,1,4)	(3,2,0)	(3,0,2)	(2,3,0)	(2,0,3)	(0,3,2)	(0,2,3)	(3,1,1)	(1,3,1)	(1,1,3)	(2,2,1)	(2,1,2)	(1,2,2)
Defender		0.000	0.000	0.000	0.008	0.027	0.000	0.000	0.000	0.029	0.177	0.098	0.101	0.135	0.095	0.127	0.072	0.061	0.069	0.000	0.000	0.000
(5,0,0)	0.000	0	3	3	3	3	3	3	-5	-5	3	3	3	3	-5	-5	-5	-5	-5	-5	-5	-5
(0,5,0)	0.000	1	0	1	1	-4	1	-4	1	1	1	-4	1	-4	1	1	-4	-4	-4	-4	-4	-4
(0,0,5)	0.000	2	2	0	-3	2	-3	2	2	2	-3	2	-3	2	2	2	-3	-3	-3	-3	-3	-3
(4,1,0)	0.000	1	3	4	0	1	3	4	-5	3	3	4	3	4	-5	-5	3	-5	3	-5	3	-5
(4,0,1)	0.077	2	5	3	2	0	5	3	3	-5	5	3	5	3	-5	-5	3	3	-5	3	-5	-5
(1,4,0)	0.000	1	3	4	1	-4	0	1	3	4	1	-4	1	-4	4	4	-4	1	1	-4	-4	1
(1,0,4)	0.057	2	5	3	-3	2	2	0	5	3	-3	2	-3	2	5	5	-3	2	2	-3	-3	2
(0,4,1)	0.000	3	2	1	3	1	2	-4	0	1	3	-4	3	-4	1	1	1	1	-4	1	-4	-4
(0,1,4)	0.000	3	2	1	2	3	-3	1	2	0	-3	3	-3	3	2	2	2	-3	2	-3	2	-3
(3,2,0)	0.140	1	3	4	1	-4	3	4	-5	4	0	1	3	4	-5	3	1	-5	4	3	4	3
(3,0,2)	0.094	2	5	3	-3	2	5	3	5	-5	2	0	5	3	3	-5	2	5	-5	5	3	3
(2,3,0)	0.041	1	3	4	1	-4	3	4	-5	4	1	-4	0	1	3	4	-4	3	4	1	1	4
(2,0,3)	0.104	2	5	3	-3	2	5	3	5	-5	-3	2	2	0	5	3	-3	5	3	2	2	5
(0,3,2)	0.000	3	2	1	3	3	-3	-4	2	1	3	1	2	-4	0	1	3	2	-4	3	1	1
(0,2,3)	0.146	3	2	1	3	3	-3	-4	2	1	2	3	-3	1	2	0	3	-3	1	2	3	2
(3,1,1)	0.067	3	5	4	2	1	5	4	3	3	2	1	5	4	-5	-5	0	3	3	3	3	-5
(1,3,1)	0.089	3	5	4	3	1	2	1	3	4	3	-4	2	-4	3	4	1	0	1	1	-4	1
(1,1,3)	0.057	3	5	4	2	3	2	1	5	3	-3	3	-3	1	5	3	2	2	0	-3	2	2
(2,2,1)	0.014	3	5	4	3	1	5	4	3	4	2	-4	2	1	-5	3	1	3	4	0	1	3
(2,1,2)	0.000	3	5	4	2	3	5	4	5	3	-3	1	2	1	3	-5	2	5	3	2	0	3
(1,2,2)	0.113	3	5	4	3	3	2	1	5	4	2	1	-3	-4	3	3	3	2	1	2	1	0

Figure 17. Illustrative instance of the General Blotto game with heterogeneous fronts ($w_i \neq w_{i+1}$) and calculated mixed strategies.

situation where one player could “win the battles but lose the war.” Given all of the available plays shown in Table 1, the mixed strategies that fall on $[0, \frac{2n}{m}]$ are $[0, \frac{10}{3}]$, or $[0, 3.\bar{3}]$, rounded to [0, 3]. Round these to the nearest integer under the rules of using integer values for x_i and y_i. These were the strategies utilized in Table 2 and Table 3. This rule will be very helpful in narrowing the feature space before formulating this game, especially after moving beyond four fronts and the theoretical permutations of resource distributions become excessively large. By thinking through the problem and making some basic assumptions on what is and is not FAA-DC, only include strategies in the feature space that are acceptable. Since this holds for asymmetric situations as well, Player 1 will only consider strategies with resource allocations, x_i, on the interval, $[0, \frac{2n_1}{m}]$, and Player 2 will only consider strategies with resource allocations, y_i, on the interval, $[0, \frac{2n_2}{m}]$.

Regardless of symmetry, assume that an attacker will have to put at least one resource on a majority of the fronts to have a chance at winning the scenario. Knowing this, the defender will also employ this strategy. This will further help narrow the feature space.

To illustrate, Figure 17 shows a normal form General Blotto game’s maximum feature space. The game is played on $m = 3$ fronts, Player 1 and Player 2 both has $n = 5$ resources. In Figure 17, all possible strategy profiles are included and Player 2 is strictly dominated by Player 1 in all profiles for $s_2 \in \{5, 0, 0\}$ because by not deploying resources to a majority of the fronts, Player 2 effectively cedes the war before “the first shot is fired.”

By making the most simple of assumptions, like Player 2 must actually attempt to win, the intervals for y_i are narrowed by restricting it so that Player 2’s pure strategies can only include those where a majority of $y_i \geq 1$. Thinking ahead to the follow-on effects of such a strategy restriction, assume that Player 1’s strategies should reflect a desire to defend against an attacker that is trying to win - so they will also employ

		Attacker																	
		(4,1,0)	(4,0,1)	(1,4,0)	(1,0,4)	(0,4,1)	(0,1,4)	(3,2,0)	(3,0,2)	(2,3,0)	(2,0,3)	(0,3,2)	(0,2,3)	(3,1,1)	(1,3,1)	(1,1,3)	(2,2,1)	(2,1,2)	(1,2,2)
Defender		0.008	0.027	0.000	0.000	0.000	0.029	0.177	0.098	0.101	0.135	0.095	0.127	0.072	0.061	0.069	0.000	0.000	0.000
(4,1,0)	0.000	0	1	3	4	-5	3	3	4	3	4	-5	-5	3	-5	3	-5	3	-5
(4,0,1)	0.077	2	0	5	3	3	-5	5	3	5	3	-5	-5	3	3	-5	3	-5	-5
(1,4,0)	0.000	1	-4	0	1	3	4	1	-4	1	-4	4	4	-4	1	1	-4	-4	1
(1,0,4)	0.057	-3	2	2	0	5	3	-3	2	-3	2	5	5	-3	2	2	-3	-3	2
(0,4,1)	0.000	3	1	2	-4	0	1	3	-4	3	-4	1	1	1	1	-4	1	-4	-4
(0,1,4)	0.000	2	3	-3	1	2	0	-3	3	-3	3	2	2	2	-3	2	-3	2	-3
(3,2,0)	0.140	1	-4	3	4	-5	4	0	1	3	4	-5	3	1	-5	4	3	4	3
(3,0,2)	0.094	-3	2	5	3	5	-5	2	0	5	3	3	-5	2	5	-5	5	3	3
(2,3,0)	0.041	1	-4	3	4	-5	4	1	-4	0	1	3	4	-4	3	4	1	1	4
(2,0,3)	0.104	-3	2	5	3	5	-5	-3	2	2	0	5	3	-3	5	3	2	2	5
(0,3,2)	0.000	3	3	-3	-4	2	1	3	1	2	-4	0	1	3	2	-4	3	1	1
(0,2,3)	0.146	3	3	-3	-4	2	1	2	3	-3	1	2	0	3	-3	1	2	3	2
(3,1,1)	0.067	2	1	5	4	3	3	2	1	5	4	-5	-5	0	3	3	3	3	-5
(1,3,1)	0.089	3	1	2	1	3	4	3	-4	2	-4	3	4	1	0	1	1	-4	1
(1,1,3)	0.057	2	3	2	1	5	3	-3	3	-3	1	5	3	2	2	0	-3	2	2
(2,2,1)	0.014	3	1	5	4	3	4	2	-4	2	1	-5	3	1	3	4	0	1	3
(2,1,2)	0.000	2	3	5	4	5	3	-3	1	2	1	3	-5	2	5	3	2	0	3
(1,2,2)	0.113	3	3	2	1	5	4	2	1	-3	-4	3	3	3	2	1	2	1	0

Figure 18. Illustrative instance of narrowing the feature space in a normal form General Blotto game.

at least one resource on a majority of the fronts. As a result, restrict Player 1's pure strategies to those where $x_i \geq 1$ for a majority of fronts. Figure 18 shows that just by thinking about what strategies are feasible, the feature space is narrowed from that in Figure 17 by over 16%.

4.3 Proof of Concept

In 1999, two Chinese Colonels, Qiao and Wang [26], outlined how a militarily inferior country such as China might counter the United States. Aside from the obvious covert methods touched upon earlier, such as: hacking into websites, targeting financial institutions, terrorism, using the media, and conducting urban warfare, the Colonels stated that weaker countries could essentially do anything: "the first rule of unrestricted warfare is that there are no rules, with nothing forbidden [26]." Seeing the world through a similar prism that Gerasimov [13] would write about 14 years later, they assert that,

> "strong countries make the rules while rising ones break them and exploit loopholes... The United States breaks [UN rules] and makes new ones

> when these rules do not suit [its purposes], but it has to observe its own rules or the whole world will not trust it [26, p. 2]."

Additionally, they note that future wars will be successful not through the skilled employment of an advanced and capable joint force, but through a new methodology that encompasses all aspects of a changing world, bringing every domain together in a common operating method [26].

The United States and Russia are similarly interested in multi-domain operations. Going beyond merely considering the Joint Force, comprising of the Services that specialize in one or more of the domains, Multi-domain Operations specifically considers the interactions and effects enabled in and between the five domains of land, sea, air, space, and cyberspace. Focusing on a multi-domain strategy ensures that all domains are covered by an element of the Joint Force, as opposed to ensuring that all Joint Force elements work together and cover each of the domains.

Set up the problem.

Through careful consideration in the setup of the model and by creating thoughtful assumptions, the General Blotto game can help map and resolve major sources of uncertainty for operational warfighting staffs' decision support scenarios. Including such a framework will help in the COA development process and further refine the JIPOE. Chinese antiaccess and multi-domain operations were introduced as a framework under which a proof-of-concept for utilizing General Blotto game in strategic planning can be set up. This thesis is not concerned with specific operational details of the People's Liberation Army (military forces of the Chinese Communist Party) (PLA) or the United States military in the Indo-Pacom Theater. Any unclassified example, while illustrative, has no real operational use, so the concern is with demonstrating the efficacy of the method and areas of improvement and additional consideration, not advocating a strategy in the Western Pacific.

The five-domain framework warrants $m = 5$ fronts. Amongst the five fronts, National (or Theater) resources or effort are distributed to overcome similar defensive resources or efforts. As shown in the illustrative instances, and as outlined in U.S. doctrine, an attacker needs many more resources than a defender to have a chance at a successful campaign. This example will utilize $n_1 = 10$ resources for the defender's resources, and $n_2 = 15$ resources for the attacker's resources. Even in these low numbers, by equation (4) the total permutations for the defender come out to $j = 1820$, and for the attacker, $k = 3876$ by equation (5).

Model the problem.

To model this proof-of concept, feature-space narrowing procedures are employed from framing the problem right through modeling and analyzing it. Fortunately, only strategies with resource allocations, x_i, on the interval, $[0, \frac{2n_1}{m}]$ or $[0, 5]$, and attacking strategies with resource allocations, y_i, on the interval, $[0, 6]$ need to be defended. In addition, only strategies where a majority (in this case, three) of the fronts have more than zero resources for both the defender and attacker need to be considered. It is assumed that all fronts have at least one resource, as the adversaries are both modern neer-peers engaging in multi-domain operations. Therefore, x_i is in the interval $[1, 5]$ and y_i is in the interval $[1, 6]$.

These parameters work out to 255 pure strategies for the defender and 591 pure strategies for the attacker. Figure 19 shows the 255 of the total 1820 Defender pure strategies and 591 of 3876 Attacker pure strategies modeled in normal form. By narrowing the feature space, only roughly 2% of the total possible space has to be modeled. The dark blue and dark red conditional formatting of the payoff matrices correspond to more decisive wins within the feature space by the defender and attacker, respectively. Of note, the other nearly 98% of the un-modeled space would be entirely blue, under this color scheme where the defender "wins" drawn or uncontested fronts.

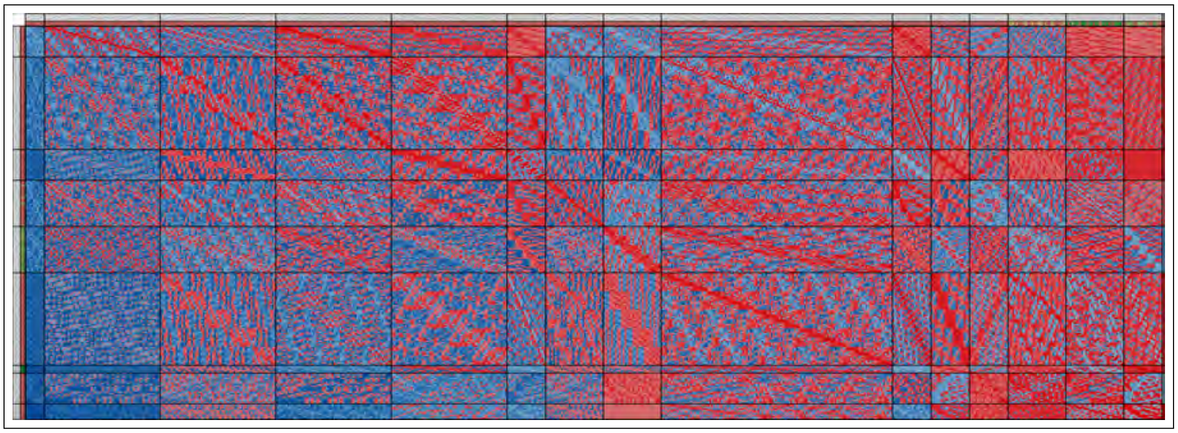


Figure 19. Low-resolution depiction of the 255 Defender pure strategies (down the left side) and 591 Attacker pure strategies (along the top) modeled in normal form. Dark blue and dark red correspond to more decisive wins by the Defender and Attacker, respectively. Please see Appendix A for a detailed version.

Upon inspection, it can be seen that the pure strategies based on permutations of the set $\{6, 6, 1, 1, 1\}$ are dominated. This is not surprising as it is assumed that all fronts would have at least one resource, and the defense wins ties, so it becomes clear that the rule that only strategies where a majority of the fronts have more than zero resources are considered does not hold for the attacker. For the attacking player, the rule should really be that a majority of the fronts should have more than one resource. By eliminating this dominated strategy, 581 strategies are left. The full graphic may be seen in Appendix A.

Analyze the problem.

This thesis assumes capabilities can be measured or assessed. Although it should also be assumed that an adversary worthy of consideration is also engaged in deception, for the purposes of planning, only incorporate known-knowns or known-unknowns. In most situations that are carefully analyzed, it is the unknown-unknowns that cause significant deviations, introduce extreme variance, or come across as "random" events. These unknown-unknowns may be unknown to both sides such as natural events or third party interference. One side's unknown-unknowns may also be results of their adversary's actions, such as deception operations, secret abilities or resources, or extreme deviations from doctrine or TTPs. Unknown-unknowns may also be any other myriad "acts of the gods."

Planning assumptions are facts that are waiting to be either proven or disproven. By modeling in assumptions, one knows what indicators to look for that may confirm or deny assumptions made. These confirmations and denials lead directly to increasing, decreasing, or completely discounting previously FAA-DC pure strategies. For every pure strategy that can be eliminated, the outcomes across all matchups with the other player are eliminated, further reducing the feature space.

This thesis assumes that the resources used in the Blotto game are use-it-or-lose it. All entities must be assigned to fronts of a particular game. Regardless of what the particular study is, all forces considered in a particular study must be utilized. Additional confirmation or denial of planning assumptions may further narrow the feature space. If intelligence narrows a range of adversarial capability on a given front, all pure strategies not conforming to this new rule can be excluded. This example already narrowed the feature space to only include pure strategies where Player 1's allocations, x_i, are on the interval, $[1, 5]$ and Player 2's allocations, y_i, are on the interval, $[1, 6]$. A further narrowing of these intervals results in the feature space being reduced significantly, providing a clearer picture of the situation.

To illustrate, in this scenario, assume Player 2 planners realize that the land, sea, and air domains (fronts 1, 2, and 3) all limit the attacker to a maximum of 4 resources due to the A2/AD strategies employed by the defender. These two strategies conversely affect the Player 1 pure strategy profile by reflecting a minimum of 2 resources in the sea and air domains (fronts 2 and 3). Removing all pure strategies that do not conform to these new restrictions means a total of 111 defensive pure strategies for Player 1 and a total of 315 offensive pure strategies (after removing the ten dominated pure strategies) for Player 2 are removed from the feature space. Following this reduction in the feature space, an additional eight Player 2 pure strategies are found to be dominated, so they are also removed.

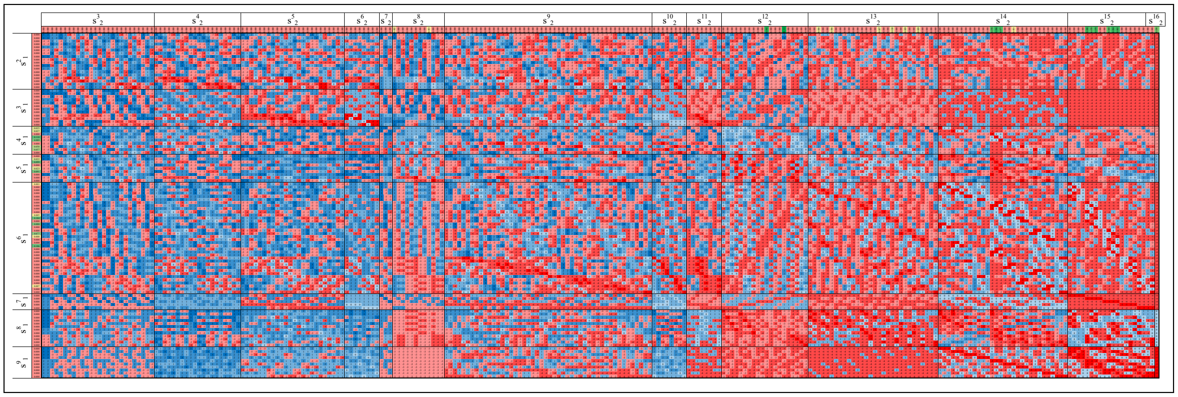


Figure 20. Low-resolution depiction of the 111 Defender pure strategies (down the left side) and 258 Attacker pure strategies (along the top) modeled in normal form. This represents a reduction of nearly 72% of the originally modeled feature space. Please see Appendix A for a detailed version.

This reduced feature space is illustrated in Figure 20. It is nearly 72% smaller than the feature space depicted in Figure 19. Any constraints such as those known in the modeling stage can easily be implemented with a linear program to create the limited pure strategies of a restricted feature space. By calculating the mixed strategies these mathematically winning strategies are always known. Preferences based on these feasible strategies begin to emerge. These strategies can be further evaluated by exception, and the assumptions made to evaluate them can be further refined or considered. With a smaller feature space, more options for analysis of prospective COAs become feasible.

4.4 Conclusion

The illustrative instances and scenario presented in this chapter show the value of creating a consolidated comparison of strategy profiles that can be narrowed as assumptions are validated. The General Blotto game helps to visualize reductions in the feature space by identifying areas of uncertainty and iteratively updating the model as the intelligence improves. The value is in understanding the strategy implications that such a game exposes. In the real world, only one round is ever played, so the strategy chosen matters. Decision points for validating assumptions (areas of uncertainty) can be identified and updated. The more branches or follow-on strategies that a commander is able to visualize, the faster they may adapt to the changing situation.

In these examples it has been shown how narrowing the probable ranges for the values of x_i and y_i has had significant impact on narrowing the feature space of possible outcomes. By setting up the model to include assumptions (uncertainty), planners can use the JPP to iteratively reduce the feature space and make the model clearer. The size and complexity of the payoff matrix is analogous to the size and uncertainty of the problem. How the strategy profiles are arranged also lends clarity to the nuances of shifting between one strategy and another - or the adversary shifting between one strategy and another. Regions in the matrix that are very similar correspond to strategy differences that do not have much expected outcome variation.

This payoff matrix can be utilized as a very high-level map where planners can quickly identify regions of interest that they want to dig into. If one wants to find a remote mountain stream, they can narrow the feature space that is a continent to the area that contains mountains and then focus on the valleys within the mountain range.

V. Conclusions and Recommendations

There is a family of imperfect numerical models reflecting alternative assumptions, objectives, and data estimates; an understanding of the solution behavior of the whole family is needed in order to fully support the development of an appropriate plan of action.

- Arthur M. Geoffrion[1]

5.1 Overview

The task of planning a theater campaign across the warfighting domains and multiple phases is a daunting one. Assessing a situation through detailed modeling and simulation is an important aspect of military planning. Quickly assessing general strategic implications of a simplified model is also important for planners and senior leaders to begin to tackle a complex problem. Utilizing the General Blotto game in the JPP, from the JIPOE through COA evaluation is helpful in framing endogenous and exogenous uncertainties. By visualizing the scope of the feature space that encompasses a planner's problem, methods to refine that space into a manageable one can be produced. By identifying, employing, and validating assumptions with the General Blotto framework, planners can effectively manage wicked problems so that their commanders can gain insight from them.

This thesis provides an example formulation with a major reduction in the feature space through just a few assumption validations. Such reductions aid planners in creating and evaluating alternative COAs. This chapter offers recommendations for further refinement of the model and recommends related areas for further research which can easily add to the realism and agility of the model.

[1]Quoted in [12, p. 81] while discussing why it is important to understand *why* a solution is optimal.

5.2 Conclusions

Success in the wars of the future requires conscious employment of interdependent operations across domains and time. The West recognizes this methodology in its likely adversaries, and those same adversaries have seen it in the actions of the West. Military doctrine and campaign planning must evolve to successfully employ these operations in the future. The General Blotto game can be manipulated to account for various endogenous and exogenous parameters required to plan for future conflict in a multi-domain operational environment.

The value of the General Blotto framework is in understanding the strategy implications that such a game exposes. In the real world, only one round is ever played, so the strategy chosen matters. The more branches or follow-on strategies that a commander is able to visualize, the faster they may adapt to the changing situation. The General Blotto game helps to visualize reductions in the feature space by identifying areas of uncertainty and iteratively updating the model as the intelligence improves. Decision points for validating assumptions (areas of uncertainty) can be identified and updated and the corresponding reduction in the feature space has been shown.

5.3 Recommendations for Future Research

Further work must be done to expand the generalizations made to the classic Colonel Blotto game. Additional research is warranted in creating utility functions to effectively compare physical and non-physical domains and effects. Spreading equivalent limited resources across fronts is essential to create a model that is meaningful and insightful.

Heterogeneous payoffs, not just between Players, but between the payoff gain and loss values for a Player in a front are an important improvement that can be made on this model. Certainly using methods other than that in equation (12) to score winning strategies could open up a whole range of new formulation possibilities. The ability to account for a situation where it is very valuable to win a front but really not that tragic to lose it would bring great agility to the generalized game. Correctly implementing heterogeneous payoffs would also allow for the modeling of situations where one player may not win a majority of the fronts, but does win the war in a utilitarian sense. A similar situation already occurs in the Electoral College in the United States. This ability would also enable better modeling of a situation such as the American Revolution or Vietnam War, where the strategies (by the United States and then against it, respectively) involved making the war as painful for the adversary as possible and continuing to fight rather than focusing on winning battles.

Going beyond this to adjusting the scoring rules so that utility is lost on lost fronts would be especially useful in modeling asymmetric conflicts where the negative impact of extended time and lack of decisive victories overwhelms the otherwise stronger player. Adjusting the victory calculations control how draws are adjudicated by front instead of by player would also be helpful.

Microsoft Excel is an effective tool, especially because of its near universal availability. Although more advanced or feature-rich programs could be created, care must be taken to avoid the temptation to create a "black box" application that loses its interactivity. Building a more advanced DSS in Visual Basic would provide near universal deployability and extended features, including a linear program for calculations and graphical setup of the bounds on the assumptions. The ability to easily test more instances faster would be a great improvement. This would also enable building much more complex instances for testing and evaluation.

Further research into the emerging literature such as Ahmadinejad, Dehghani, Hajiaghayi, Lucier, Mahini, and Seddighin's [1] to create agile algorithms to quickly calculate the more complex General Blotto formulations could be helpful.

5.4 Summary

This research utilizes a generalized version of the classic Colonel Blotto game to understand adversarial equilibriums and identify areas of opportunity or vulnerability with the intent that further detailed assessment is required. The intent is not to replace full-scale, detailed models, but for a simplified auxiliary model to supplement them [12].

In "Antiaccess Warfare as Strategy," [40] Tangredi posits the question and need to consider multiple domains and governmental and warfighting functions in various phases of campaign execution. Multi-domain integration within and across various phases of the joint campaign presents a host of non-linear factors that are compounded and amplified by uncertainties.

This thesis devises and demonstrates a suitable proof-of-concept that helpes to identify, expose, and resolve some sources of uncertainty in a decision support scenario. This research explores some of the formulation schema, data-driven parameters, methods of calculation, and scenarios applicable to the design and construction of the General Blotto game and demonstrates its utility to future campaign phase planning and time-sensitive operational environments.

Appendix A. Excel Sheets - Inputs and Payoff Matrices

Illustrative Instances.

Asymmetric vs. Symmetric Forces.

Microsoft Excel data for creating the $m = 2$ fronts, $n = 4$ resources, symmetric illustrative instance payoff matrix:

Defender Strategies — Armies: 4, Fronts: 2

5 total strategy permutations

#	Strategy	Permutation			Front 1 weight: 1	Front 2 weight: 1
1	01	A	DS01A	(4,0)	4	0
2	01	B	DS01B	(0,4)	0	4
3	02	A	DS02A	(3,1)	3	1
4	02	B	DS02B	(1,3)	1	3
5	03		DS03	(2,2)	2	2

Attacker Strategies — Armies: 4, Fronts: 2

5 total strategy permutations

#	Strategy	Permutation			Front 1 weight: 1	Front 2 weight: 1
1	01	A	AS01A	(4,0)	4	0
2	01	B	AS01B	(0,4)	0	4
3	02	A	AS02A	(3,1)	3	1
4	02	B	AS02B	(1,3)	1	3
5	03		AS03	(2,2)	2	2

Conditionally formatted, illustrative instance, payoff matrix for the symmetric game of $m = 2$ fronts and $n = 4$ resources:

		Attacker				
		(4,0)	(0,4)	(3,1)	(1,3)	(2,2)
Defender		0.200	0.200	0.200	0.200	0.200
(4,0)	0.200	0	1	1	1	1
(0,4)	0.200	1	0	1	1	1
(3,1)	0.200	1	1	0	1	1
(1,3)	0.200	1	1	1	0	1
(2,2)	0.200	1	1	1	1	0

Microsoft Excel data for creating the $m = 2$ fronts, $n_1 = 4$ and $n_2 = 6$ resources, asymmetric illustrative instance payoff matrix:

Defender Strategies — Armies: 4, Fronts: 2

5 total strategy permutations

#	Strategy	Permutation			Front 1 weight: 1	Front 2 weight: 1
1	01	A	DS01A	(4,0)	4	0
2	01	B	DS01B	(0,4)	0	4
3	02	A	DS02A	(3,1)	3	1
4	02	B	DS02B	(1,3)	1	3
5	03	A	DS03A	(2,2)	2	2

Attacker Strategies — Armies: 6, Fronts: 2

7 total strategy permutations

#	Strategy	Permutation			Front 1 weight: 1	Front 2 weight: 1
1	01	A	AS01A	(6,0)	6	0
2	01	B	AS01B	(0,6)	0	6
3	02	A	AS02A	(5,1)	5	1
4	02	B	AS02B	(1,5)	1	5
5	03	A	AS03A	(4,2)	4	2
6	03	B	AS03B	(2,4)	2	4
7	04		AS04	(3,3)	3	3

Conditionally formatted, illustrative instance, payoff matrix for the asymmetric game of $m = 2$ fronts and $n_1 = 4$ and $n_2 = 6$ resources:

		Attacker						
		(6,0)	(0,6)	(5,1)	(1,5)	(4,2)	(2,4)	(3,3)
Defender		0.000	0.000	0.250	0.250	0.150	0.150	0.200
(4,0)	0.250	0	1	-2	1	0	1	1
(0,4)	0.250	1	0	1	-2	1	0	1
(3,1)	0.150	1	1	0	1	-2	1	0
(1,3)	0.150	1	1	1	0	1	-2	0
(2,2)	0.200	1	1	1	1	0	0	-2

Increasing the Fronts.

Microsoft Excel data for creating the $m = 2$ fronts, $n_1 = 5$ and $n_2 = 4$ resources, illustrative instance payoff matrix:

					Front 1	Front 2
	Defender Strategies		Armies:	5		
			Fronts:	2	Front 1	Front 2
6	total strategy permutations				weight:	weight:
#	Strategy	Permutation			1	1
1	01	A	DS01A	(5,0)	5	0
2	01	B	DS01B	(0,5)	0	5
3	02	A	DS02A	(4,1)	4	1
4	02	B	DS02B	(1,4)	1	4
5	03	A	DS03A	(3,2)	3	2
6	03	B	DS03B	(2,3)	2	3

	Attacker Strategies		Armies:	4		
			Fronts:	2	Front 1	Front 2
5	total strategy permutations				weight:	weight:
#	Strategy	Permutation			1	1
1	01	A	AS01A	(4,0)	4	0
2	01	B	AS01B	(0,4)	0	4
3	02	A	AS02A	(3,1)	3	1
4	02	B	AS02B	(1,3)	1	3
5	03		AS03	(2,2)	2	2

Conditionally formatted, illustrative instance, payoff matrix for the game of $m = 2$ fronts and $n_1 = 5$ and $n_2 = 4$ resources:

		Attacker				
		(4,0)	(0,4)	(3,1)	(1,3)	(2,2)
Defender		0.200	0.200	0.200	0.200	0.200
(5,0)	0.167	1	1	1	1	1
(0,5)	0.167	1	1	1	1	1
(4,1)	0.167	1	1	1	1	1
(1,4)	0.167	1	1	1	1	1
(3,2)	0.167	1	1	1	1	1
(2,3)	0.167	1	1	1	1	1

Microsoft Excel data for Player 2 to open an additional front to have a total of $m = 3$ fronts, the same $n_1 = 5$ and $n_2 = 4$ resources, illustrative instance payoff matrix:

Defender Strategies			Armies:	5			
			Fronts:	3	Front 1	Front 2	Front 3
21	total strategy permutations				weight:	weight:	weight:
#	Strategy	Permutation			1	1	1
1	01	A	DS01A	(5,0,0)	5	0	0
2	01	B	DS01B	(0,5,0)	0	5	0
3	01	C	DS01C	(0,0,5)	0	0	5
4	02	A	DS02A	(4,1,0)	4	1	0
5	02	B	DS02B	(4,0,1)	4	0	1
6	02	C	DS02C	(1,4,0)	1	4	0
7	02	D	DS02D	(1,0,4)	1	0	4
8	02	E	DS02E	(0,4,1)	0	4	1
9	02	F	DS02F	(0,1,4)	0	1	4
10	03	A	DS03A	(3,2,0)	3	2	0
11	03	B	DS03B	(3,0,2)	3	0	2
12	03	C	DS03C	(2,3,0)	2	3	0
13	03	D	DS03D	(2,0,3)	2	0	3
14	03	E	DS03E	(0,3,2)	0	3	2
15	03	F	DS03F	(0,2,3)	0	2	3
16	04	A	DS04A	(3,1,1)	3	1	1
17	04	B	DS04B	(1,3,1)	1	3	1
18	04	C	DS04C	(1,1,3)	1	1	3
19	05	A	DS05A	(2,2,1)	2	2	1
20	05	B	DS05B	(2,1,2)	2	1	2
21	05	C	DS05C	(1,2,2)	1	2	2

Attacker Strategies			Armies:	4			
			Fronts:	3	Front 1	Front 2	Front 3
15	total strategy permutations				weight:	weight:	weight:
#	Strategy	Permutation			1	1	1
1	01	A	AS01A	(4,0,0)	4	0	0
2	01	B	AS01B	(0,4,0)	0	4	0
3	01	C	AS01C	(0,0,4)	0	0	4
4	02	A	AS02A	(3,1,0)	3	1	0
5	02	B	AS02B	(3,0,1)	3	0	1
6	02	C	AS02C	(1,3,0)	1	3	0
7	02	D	AS02D	(1,0,3)	1	0	3
8	02	E	AS02E	(0,3,1)	0	3	1
9	02	F	AS02F	(0,1,3)	0	1	3
10	03	A	AS03A	(2,2,0)	2	2	0
11	03	B	AS03B	(2,0,2)	2	0	2
12	03	C	AS03C	(0,2,2)	0	2	2
13	04	A	AS04A	(2,1,1)	2	1	1
14	04	B	AS04B	(1,2,1)	1	2	1
15	04	C	AS04C	(1,1,2)	1	1	2

Conditionally formatted, illustrative instance, payoff matrix for the game where Player 2 opened an additional front to have a total of $m = 3$ fronts with the same $n_1 = 5$ and $n_2 = 4$ resources:

		Attacker														
		(4,0,0)	(0,4,0)	(0,0,4)	(3,1,0)	(3,0,1)	(1,3,0)	(1,0,3)	(0,3,1)	(0,1,3)	(2,2,0)	(2,0,2)	(0,2,2)	(2,1,1)	(1,2,1)	(1,1,2)
Defender		0.000	0.000	0.000	0.032	0.032	0.048	0.048	0.063	0.063	0.269	0.269	0.176	0.000	0.000	0.000
(5,0,0)	0.000	1	1	1	1	1	1	1	-2	-2	1	1	-2	-2	-2	-2
(0,5,0)	0.000	1	1	1	1	-2	1	-2	1	1	1	-2	1	-2	-2	-2
(0,0,5)	0.000	1	1	1	-2	1	-2	1	1	1	-2	1	1	-2	-2	-2
(4,1,0)	0.000	1	1	2	1	2	1	2	-2	1	1	2	-2	1	-2	1
(4,0,1)	0.000	1	2	1	2	1	2	1	1	-2	2	1	-2	1	1	-2
(1,4,0)	0.000	1	1	2	1	-2	1	1	2	2	1	-2	2	-2	1	1
(1,0,4)	0.000	1	2	1	-2	1	1	1	2	2	-2	1	2	-2	1	1
(0,4,1)	0.000	2	1	1	2	1	2	-2	1	1	2	-2	1	1	1	-2
(0,1,4)	0.000	2	1	1	1	2	-2	2	1	1	-2	2	1	1	-2	1
(3,2,0)	0.167	1	1	2	1	1	1	2	-2	2	1	2	1	2	1	2
(3,0,2)	0.167	1	2	1	1	1	2	1	2	-2	2	1	1	2	2	1
(2,3,0)	0.095	1	1	2	1	-2	1	2	1	2	1	1	2	1	2	2
(2,0,3)	0.095	1	2	1	-2	1	2	1	2	1	1	1	2	1	2	2
(0,3,2)	0.024	2	1	1	2	2	1	-2	1	1	2	1	1	2	2	1
(0,2,3)	0.024	2	1	1	2	2	-2	1	1	1	1	2	1	2	1	2
(3,1,1)	0.000	2	2	2	1	1	2	2	1	1	2	2	-2	1	1	1
(1,3,1)	0.000	2	2	2	2	1	1	1	1	2	2	-2	2	1	1	1
(1,1,3)	0.000	2	2	2	1	2	1	1	2	1	-2	2	2	1	1	1
(2,2,1)	0.000	2	2	2	2	1	2	2	1	2	1	1	1	1	1	2
(2,1,2)	0.000	2	2	2	1	2	2	2	2	1	1	1	1	1	2	1
(1,2,2)	0.429	2	2	2	2	2	1	1	2	2	1	1	1	2	1	1

Homogeneous vs. Heterogeneous Fronts.

Microsoft Excel data for creating the $m = 3$ fronts, the same $n = 5$ resources, illustrative instance payoff matrix with homogeneous fronts:

Defender Strategies			Armies:	5			
			Fronts:	3	Front 1	Front 2	Front 3
21	total strategy permutations				weight:	weight:	weight:
#	Strategy	Permutation			1	1	1
1	01	A	DS01A	(5,0,0)	5	0	0
2	01	B	DS01B	(0,5,0)	0	5	0
3	01	C	DS01C	(0,0,5)	0	0	5
4	02	A	DS02A	(4,1,0)	4	1	0
5	02	B	DS02B	(4,0,1)	4	0	1
6	02	C	DS02C	(1,4,0)	1	4	0
7	02	D	DS02D	(1,0,4)	1	0	4
8	02	E	DS02E	(0,4,1)	0	4	1
9	02	F	DS02F	(0,1,4)	0	1	4
10	03	A	DS03A	(3,2,0)	3	2	0
11	03	B	DS03B	(3,0,2)	3	0	2
12	03	C	DS03C	(2,3,0)	2	3	0
13	03	D	DS03D	(2,0,3)	2	0	3
14	03	E	DS03E	(0,3,2)	0	3	2
15	03	F	DS03F	(0,2,3)	0	2	3
16	04	A	DS04A	(3,1,1)	3	1	1
17	04	B	DS04B	(1,3,1)	1	3	1
18	04	C	DS04C	(1,1,3)	1	1	3
19	05	A	DS05A	(2,2,1)	2	2	1
20	05	B	DS05B	(2,1,2)	2	1	2
21	05	C	DS05C	(1,2,2)	1	2	2

Attacker Strategies			Armies:	5			
			Fronts:	3	Front 1	Front 2	Front 3
21	total strategy permutations				weight:	weight:	weight:
#	Strategy	Permutation			1	1	1
1	01	A	AS01A	(5,0,0)	5	0	0
2	01	B	AS01B	(0,5,0)	0	5	0
3	01	C	AS01C	(0,0,5)	0	0	5
4	02	A	AS02A	(4,1,0)	4	1	0
5	02	B	AS02B	(4,0,1)	4	0	1
6	02	C	AS02C	(1,4,0)	1	4	0
7	02	D	AS02D	(1,0,4)	1	0	4
8	02	E	AS02E	(0,4,1)	0	4	1
9	02	F	AS02F	(0,1,4)	0	1	4
10	03	A	AS03A	(3,2,0)	3	2	0
11	03	B	AS03B	(3,0,2)	3	0	2
12	03	C	AS03C	(2,3,0)	2	3	0
13	03	D	AS03D	(2,0,3)	2	0	3
14	03	E	AS03E	(0,3,2)	0	3	2
15	03	F	AS03F	(0,2,3)	0	2	3
16	04	A	AS04A	(3,1,1)	3	1	1
17	04	B	AS04B	(1,3,1)	1	3	1
18	04	C	AS04C	(1,1,3)	1	1	3
19	05	A	AS05A	(2,2,1)	2	2	1
20	05	B	AS05B	(2,1,2)	2	1	2
21	05	C	AS05C	(1,2,2)	1	2	2

Conditionally formatted, illustrative instance, payoff matrix for the game with $m = 3$ homogeneous fronts and $n = 5$ resources:

		Attacker																				
		(5,0,0)	(0,5,0)	(0,0,5)	(4,1,0)	(4,0,1)	(1,4,0)	(1,0,4)	(0,4,1)	(0,1,4)	(3,2,0)	(3,0,2)	(2,3,0)	(2,0,3)	(0,3,2)	(0,2,3)	(3,1,1)	(1,3,1)	(1,1,3)	(2,2,1)	(2,1,2)	(1,2,2)
Defender		0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.111	0.111	0.111	0.111	0.111	0.111	0.111	0.111	0.111	0.000	0.000	0.000
(5,0,0)	0.000	0	1	1	1	1	1	1	-2	-2	1	1	1	1	-2	-2	-2	-2	-2	-2	-2	-2
(0,5,0)	0.000	1	0	1	1	-2	1	-2	1	1	1	-2	1	-2	1	1	-2	-2	-2	-2	-2	-2
(0,0,5)	0.000	1	1	0	-2	1	-2	1	1	1	-2	1	-2	1	1	1	-2	-2	-2	-2	-2	-2
(4,1,0)	0.000	1	1	2	0	1	1	2	-2	1	1	2	1	2	-2	-2	1	-2	1	-2	1	-2
(4,0,1)	0.000	1	2	1	1	0	2	1	1	-2	2	1	2	1	-2	-2	1	1	-2	1	-2	-2
(1,4,0)	0.000	1	1	2	1	-2	0	1	1	2	1	-2	1	-2	2	2	-2	1	1	-2	-2	1
(1,0,4)	0.000	1	2	1	-2	1	1	0	2	1	-2	1	-2	1	2	2	-2	1	1	-2	-2	1
(0,4,1)	0.000	2	1	1	2	1	1	-2	0	1	2	-2	2	-2	1	1	1	1	-2	1	-2	-2
(0,1,4)	0.000	2	1	1	1	2	-2	1	1	0	-2	2	-2	2	1	1	1	-2	1	-2	1	-2
(3,2,0)	0.111	1	1	2	1	-2	1	2	-2	2	0	1	1	2	-2	1	1	-2	2	1	2	1
(3,0,2)	0.111	1	2	1	-2	1	2	1	2	-2	1	0	2	1	1	-2	1	2	-2	2	1	1
(2,3,0)	0.111	1	1	2	1	-2	1	2	-2	2	1	-2	0	1	1	2	-2	1	2	1	1	2
(2,0,3)	0.111	1	2	1	-2	1	2	1	2	-2	-2	1	1	0	2	1	-2	2	1	1	1	2
(0,3,2)	0.111	2	1	1	2	2	-2	-2	1	1	2	1	1	-2	0	1	2	1	-2	2	1	1
(0,2,3)	0.111	2	1	1	2	2	-2	-2	1	1	1	2	-2	1	1	0	2	-2	1	1	2	1
(3,1,1)	0.111	2	2	2	1	1	2	2	1	1	1	1	2	2	-2	-2	0	1	1	1	1	-2
(1,3,1)	0.111	2	2	2	2	1	1	1	1	2	2	-2	1	-2	1	2	1	0	1	1	-2	1
(1,1,3)	0.111	2	2	2	1	2	1	1	2	1	-2	2	-2	1	2	1	1	1	0	-2	1	1
(2,2,1)	0.000	2	2	2	2	1	2	2	1	2	1	-2	1	1	-2	1	1	1	2	0	1	1
(2,1,2)	0.000	2	2	2	1	2	2	2	2	1	-2	1	1	1	1	-2	1	2	1	1	0	1
(1,2,2)	0.000	2	2	2	2	2	1	1	2	2	1	1	-2	-2	1	1	2	1	1	1	1	0

Microsoft Excel data for creating the $m = 3$ fronts, the same $n = 5$ resources, illustrative instance payoff matrix with heterogeneous fronts:

Defender Strategies			Armies:	5			
			Fronts:	3	Front 1	Front 2	Front 3
21	total strategy permutations				weight:	weight:	weight:
#	Strategy	Permutation			3	1	2
1	01	A	DS01A	(5,0,0)	5	0	0
2	01	B	DS01B	(0,5,0)	0	5	0
3	01	C	DS01C	(0,0,5)	0	0	5
4	02	A	DS02A	(4,1,0)	4	1	0
5	02	B	DS02B	(4,0,1)	4	0	1
6	02	C	DS02C	(1,4,0)	1	4	0
7	02	D	DS02D	(1,0,4)	1	0	4
8	02	E	DS02E	(0,4,1)	0	4	1
9	02	F	DS02F	(0,1,4)	0	1	4
10	03	A	DS03A	(3,2,0)	3	2	0
11	03	B	DS03B	(3,0,2)	3	0	2
12	03	C	DS03C	(2,3,0)	2	3	0
13	03	D	DS03D	(2,0,3)	2	0	3
14	03	E	DS03E	(0,3,2)	0	3	2
15	03	F	DS03F	(0,2,3)	0	2	3
16	04	A	DS04A	(3,1,1)	3	1	1
17	04	B	DS04B	(1,3,1)	1	3	1
18	04	C	DS04C	(1,1,3)	1	1	3
19	05	A	DS05A	(2,2,1)	2	2	1
20	05	B	DS05B	(2,1,2)	2	1	2
21	05	C	DS05C	(1,2,2)	1	2	2

Attacker Strategies			Armies:	5			
			Fronts:	3	Front 1	Front 2	Front 3
21	total strategy permutations				weight:	weight:	weight:
#	Strategy	Permutation			1	2	3
1	01	A	AS01A	(5,0,0)	5	0	0
2	01	B	AS01B	(0,5,0)	0	5	0
3	01	C	AS01C	(0,0,5)	0	0	5
4	02	A	AS02A	(4,1,0)	4	1	0
5	02	B	AS02B	(4,0,1)	4	0	1
6	02	C	AS02C	(1,4,0)	1	4	0
7	02	D	AS02D	(1,0,4)	1	0	4
8	02	E	AS02E	(0,4,1)	0	4	1
9	02	F	AS02F	(0,1,4)	0	1	4
10	03	A	AS03A	(3,2,0)	3	2	0
11	03	B	AS03B	(3,0,2)	3	0	2
12	03	C	AS03C	(2,3,0)	2	3	0
13	03	D	AS03D	(2,0,3)	2	0	3
14	03	E	AS03E	(0,3,2)	0	3	2
15	03	F	AS03F	(0,2,3)	0	2	3
16	04	A	AS04A	(3,1,1)	3	1	1
17	04	B	AS04B	(1,3,1)	1	3	1
18	04	C	AS04C	(1,1,3)	1	1	3
19	05	A	AS05A	(2,2,1)	2	2	1
20	05	B	AS05B	(2,1,2)	2	1	2
21	05	C	AS05C	(1,2,2)	1	2	2

Conditionally formatted, illustrative instance, payoff matrix for the game with $m = 3$ heterogeneous fronts and $n = 5$ resources:

		Attacker																				
		(5,0,0)	(0,5,0)	(0,0,5)	(4,1,0)	(4,0,1)	(1,4,0)	(1,0,4)	(0,4,1)	(0,1,4)	(3,2,0)	(3,0,2)	(2,3,0)	(2,0,3)	(0,3,2)	(0,2,3)	(3,1,1)	(1,3,1)	(1,1,3)	(2,2,1)	(2,1,2)	(1,2,2)
Defender		0.000	0.000	0.000	0.008	0.027	0.000	0.000	0.000	0.029	0.177	0.098	0.101	0.135	0.095	0.127	0.072	0.061	0.069	0.000	0.000	0.000
(5,0,0)	0.000	0	3	3	3	3	3	3	-5	-5	3	3	3	3	-5	-5	-5	-5	-5	-5	-5	-5
(0,5,0)	0.000	1	0	1	1	-4	1	-4	1	1	1	-4	1	-4	1	1	-4	-4	-4	-4	-4	-4
(0,0,5)	0.000	2	2	0	-3	2	-3	2	2	2	-3	2	-3	2	2	2	-3	-3	-3	-3	-3	-3
(4,1,0)	0.000	1	3	4	0	1	3	4	-5	3	3	4	3	4	-5	-5	3	-5	3	-5	3	-5
(4,0,1)	0.077	2	5	3	2	0	5	3	3	-5	5	3	5	3	-5	-5	3	3	-5	3	-5	-5
(1,4,0)	0.000	1	3	4	1	-4	0	1	3	4	1	-4	1	-4	4	4	-4	1	1	-4	-4	1
(1,0,4)	0.057	2	5	3	-3	2	2	0	5	3	-3	2	-3	2	5	5	-3	2	2	-3	-3	2
(0,4,1)	0.000	3	2	1	3	1	2	-4	0	1	3	-4	3	-4	1	1	1	1	-4	1	-4	-4
(0,1,4)	0.000	3	2	1	2	3	-3	1	2	0	-3	3	-3	3	2	2	2	-3	2	-3	2	-3
(3,2,0)	0.140	1	3	4	1	-4	3	4	-5	4	0	1	3	4	-5	3	1	-5	4	3	4	3
(3,0,2)	0.094	2	5	3	-3	2	5	3	5	-5	2	0	5	3	3	-5	2	5	-5	5	3	3
(2,3,0)	0.041	1	3	4	1	-4	3	4	-5	4	1	-4	0	1	3	4	-4	3	4	1	1	4
(2,0,3)	0.104	2	5	3	-3	2	5	3	5	-5	-3	2	2	0	5	3	-3	5	3	2	2	5
(0,3,2)	0.000	3	2	1	3	3	-3	-4	2	1	3	1	2	-4	0	1	3	2	-4	3	1	1
(0,2,3)	0.146	3	2	1	3	3	-3	-4	2	1	2	3	-3	1	2	0	3	-3	1	2	3	2
(3,1,1)	0.067	3	5	4	2	1	5	4	3	3	2	1	5	4	-5	-5	0	3	3	3	3	-5
(1,3,1)	0.089	3	5	4	3	1	2	1	3	4	3	-4	2	-4	3	4	1	0	1	1	-4	1
(1,1,3)	0.057	3	5	4	2	3	2	1	5	3	-3	3	-3	1	5	3	2	2	0	-3	2	2
(2,2,1)	0.014	3	5	4	3	1	5	4	3	4	2	-4	2	1	-5	3	1	3	4	0	1	3
(2,1,2)	0.000	3	5	4	2	3	5	4	5	3	-3	1	2	1	3	-5	2	5	3	2	0	3
(1,2,2)	0.113	3	5	4	3	3	2	1	5	4	2	1	-3	-4	3	3	3	2	1	2	1	0

Narrowing the Feature Space.

Microsoft Excel data for creating the $m = 3$ fronts, the same $n = 5$ resources, illustrative instance payoff matrix where strategy profiles have been removed to narrow the feature space:

#	Defender Strategies				Front 1	Front 2	Front 3
	Armies: 5						
	Fronts: 3						
21	total strategy permutations				weight: 3	weight: 1	weight: 2
#	Strategy	Permutation					
4	02	A	DS02A	(4,1,0)	4	1	0
5	02	B	DS02B	(4,0,1)	4	0	1
6	02	C	DS02C	(1,4,0)	1	4	0
7	02	D	DS02D	(1,0,4)	1	0	4
8	02	E	DS02E	(0,4,1)	0	4	1
9	02	F	DS02F	(0,1,4)	0	1	4
10	03	A	DS03A	(3,2,0)	3	2	0
11	03	B	DS03B	(3,0,2)	3	0	2
12	03	C	DS03C	(2,3,0)	2	3	0
13	03	D	DS03D	(2,0,3)	2	0	3
14	03	E	DS03E	(0,3,2)	0	3	2
15	03	F	DS03F	(0,2,3)	0	2	3
16	04	A	DS04A	(3,1,1)	3	1	1
17	04	B	DS04B	(1,3,1)	1	3	1
18	04	C	DS04C	(1,1,3)	1	1	3
19	05	A	DS05A	(2,2,1)	2	2	1
20	05	B	DS05B	(2,1,2)	2	1	2
21	05	C	DS05C	(1,2,2)	1	2	2

#	Attacker Strategies				Front 1	Front 2	Front 3
	Armies: 5						
	Fronts: 3						
21	total strategy permutations				weight: 1	weight: 2	weight: 3
#	Strategy	Permutation					
4	02	A	AS02A	(4,1,0)	4	1	0
5	02	B	AS02B	(4,0,1)	4	0	1
6	02	C	AS02C	(1,4,0)	1	4	0
7	02	D	AS02D	(1,0,4)	1	0	4
8	02	E	AS02E	(0,4,1)	0	4	1
9	02	F	AS02F	(0,1,4)	0	1	4
10	03	A	AS03A	(3,2,0)	3	2	0
11	03	B	AS03B	(3,0,2)	3	0	2
12	03	C	AS03C	(2,3,0)	2	3	0
13	03	D	AS03D	(2,0,3)	2	0	3
14	03	E	AS03E	(0,3,2)	0	3	2
15	03	F	AS03F	(0,2,3)	0	2	3
16	04	A	AS04A	(3,1,1)	3	1	1
17	04	B	AS04B	(1,3,1)	1	3	1
18	04	C	AS04C	(1,1,3)	1	1	3
19	05	A	AS05A	(2,2,1)	2	2	1
20	05	B	AS05B	(2,1,2)	2	1	2
21	05	C	AS05C	(1,2,2)	1	2	2

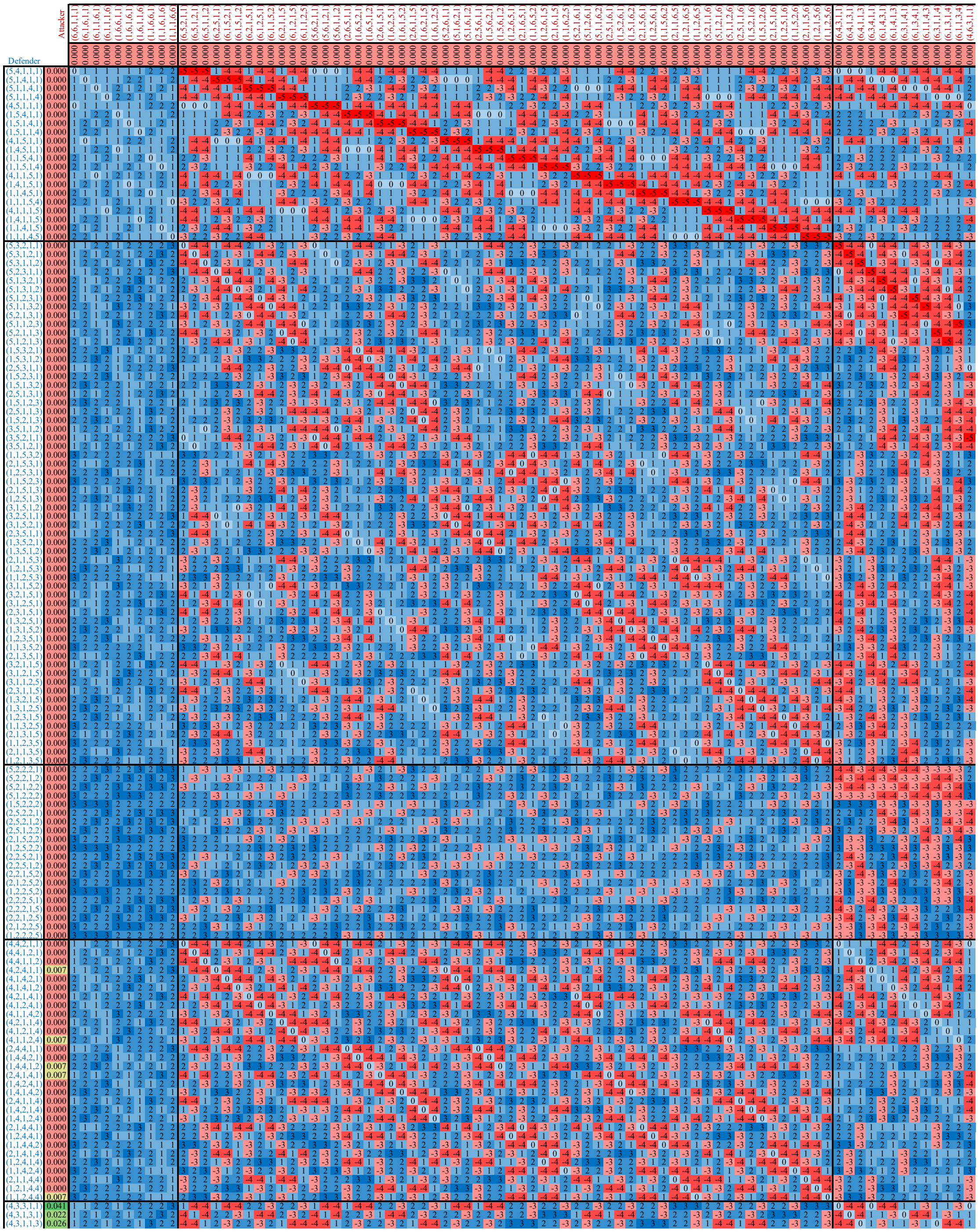
Conditionally formatted, illustrative instance, payoff matrix for the game with $m = 3$ and $n = 5$ resources where strategy profiles have been removed to narrow the feature space:

		Attacker																	
		(4,1,0)	(4,0,1)	(1,4,0)	(1,0,4)	(0,4,1)	(0,1,4)	(3,2,0)	(3,0,2)	(2,3,0)	(2,0,3)	(0,3,2)	(0,2,3)	(3,1,1)	(1,3,1)	(1,1,3)	(2,2,1)	(2,1,2)	(1,2,2)
Defender		0.008	0.027	0.000	0.000	0.000	0.029	0.177	0.098	0.101	0.135	0.095	0.127	0.072	0.061	0.069	0.000	0.000	0.000
(4,1,0)	0.000	0	1	3	4	-5	3	3	4	3	4	-5	-5	3	-5	3	-5	3	-5
(4,0,1)	0.077	2	0	5	3	3	-5	5	3	5	3	-5	-5	3	3	-5	3	-5	-5
(1,4,0)	0.000	1	-4	0	1	3	4	1	-4	1	-4	4	4	-4	1	1	-4	-4	1
(1,0,4)	0.057	-3	2	2	0	5	3	-3	2	-3	2	5	5	-3	2	2	-3	-3	2
(0,4,1)	0.000	3	1	2	-4	0	1	3	-4	3	-4	1	1	1	1	-4	1	-4	-4
(0,1,4)	0.000	2	3	-3	1	2	0	-3	3	-3	3	2	2	2	-3	2	-3	2	-3
(3,2,0)	0.140	1	-4	3	4	-5	4	0	1	3	4	-5	3	1	-5	4	3	4	3
(3,0,2)	0.094	-3	2	5	3	5	-5	2	0	5	3	3	-5	2	5	-5	5	3	3
(2,3,0)	0.041	1	-4	3	4	-5	4	1	-4	0	1	3	4	-4	3	4	1	1	4
(2,0,3)	0.104	-3	2	5	3	5	-5	-3	2	2	0	5	3	-3	5	3	2	2	5
(0,3,2)	0.000	3	3	-3	-4	2	1	3	1	2	-4	0	1	3	2	-4	3	1	1
(0,2,3)	0.146	3	3	-3	-4	2	1	2	3	-3	1	2	0	3	-3	1	2	3	2
(3,1,1)	0.067	2	1	5	4	3	3	2	1	5	4	-5	-5	0	3	3	3	3	-5
(1,3,1)	0.089	3	1	2	1	3	4	3	-4	2	-4	3	4	1	0	1	1	-4	1
(1,1,3)	0.057	2	3	2	1	5	3	-3	3	-3	1	5	3	2	2	0	-3	2	2
(2,2,1)	0.014	3	1	5	4	3	4	2	-4	2	1	-5	3	1	3	4	0	1	3
(2,1,2)	0.000	2	3	5	4	5	3	-3	1	2	1	3	-5	2	5	3	2	0	3
(1,2,2)	0.113	3	3	2	1	5	4	2	1	-3	-4	3	3	3	2	1	2	1	0

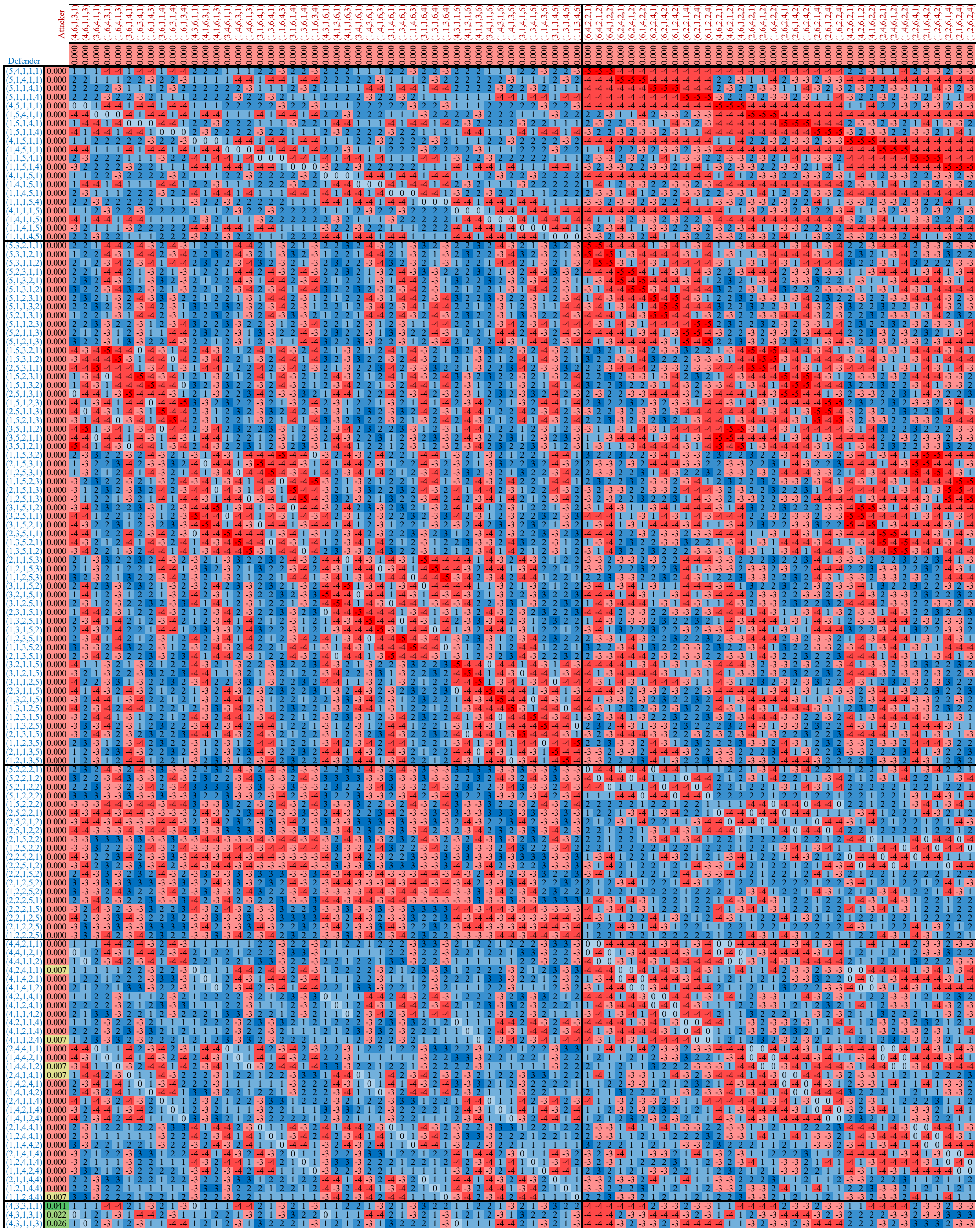
Proof of Concept.

First model.

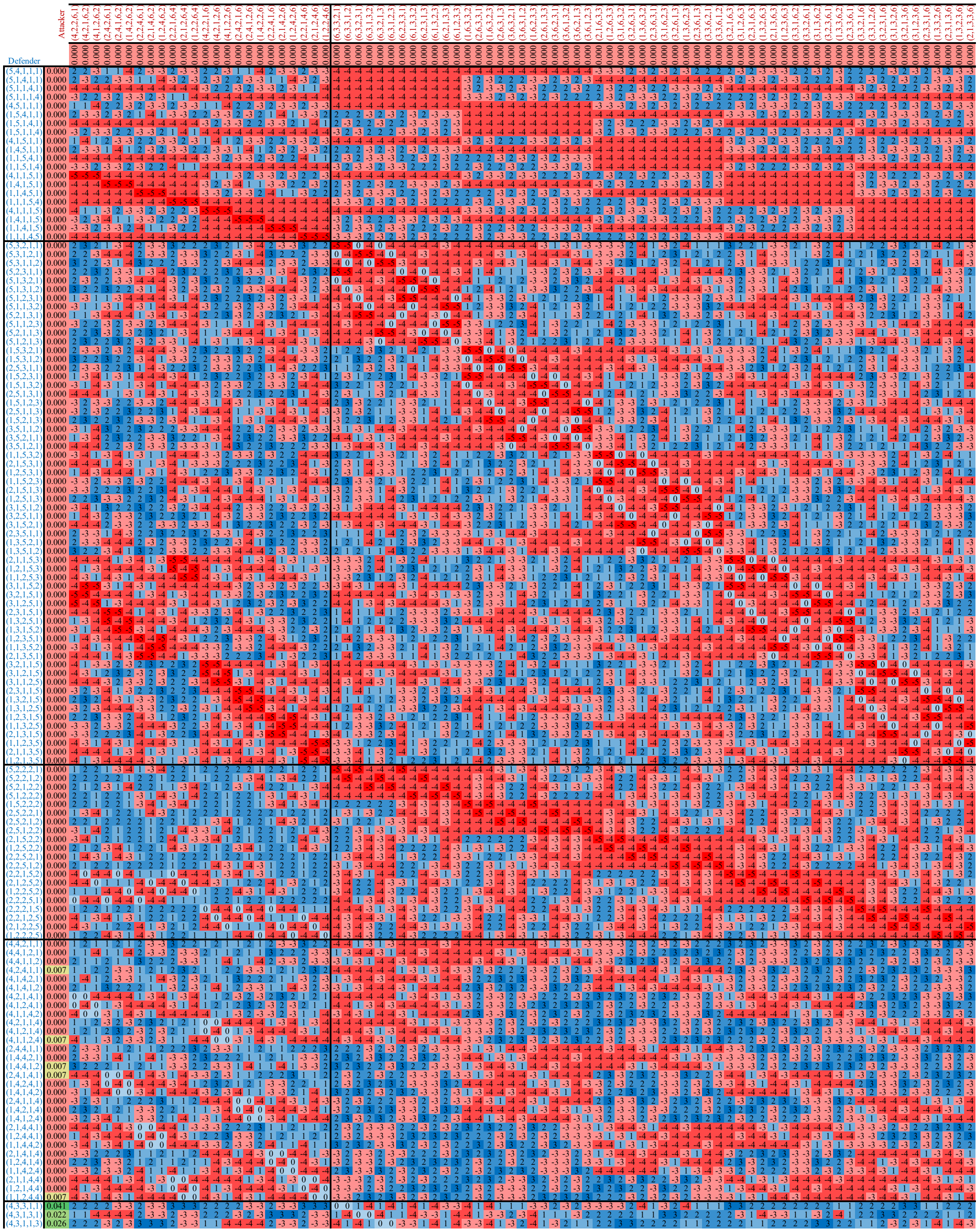
A detailed version of the first model, depicted in low-resolution in Figure 19 on page 68, follows. These 16 sections of the payoff matrix comprise the full matrix depicted in Figure 19. Although difficult to read in printed form, the electronic version of this thesis contains all of the matrix data and can be much more easily read. The first eight make up the top half of 19, while the second eight make up the bottom half of the matrix. Based on the feature space reduction methods employed in the setup, this output is a payoff matrix of size $j = 255$ pure strategies and $k = 591$ pure strategies. They are labeled by pure strategy and include the mixed strategies for the game.



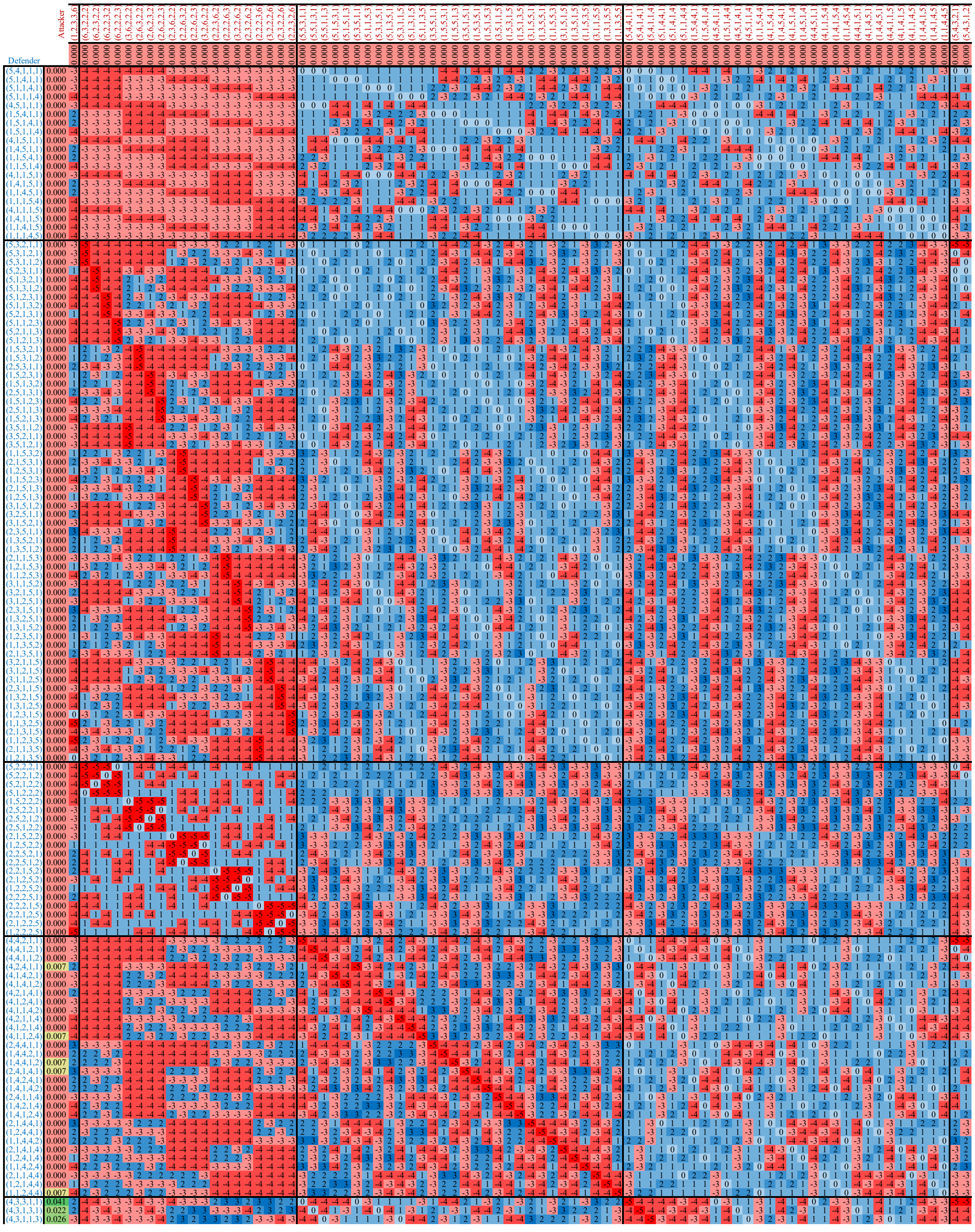
Section 1 of 16



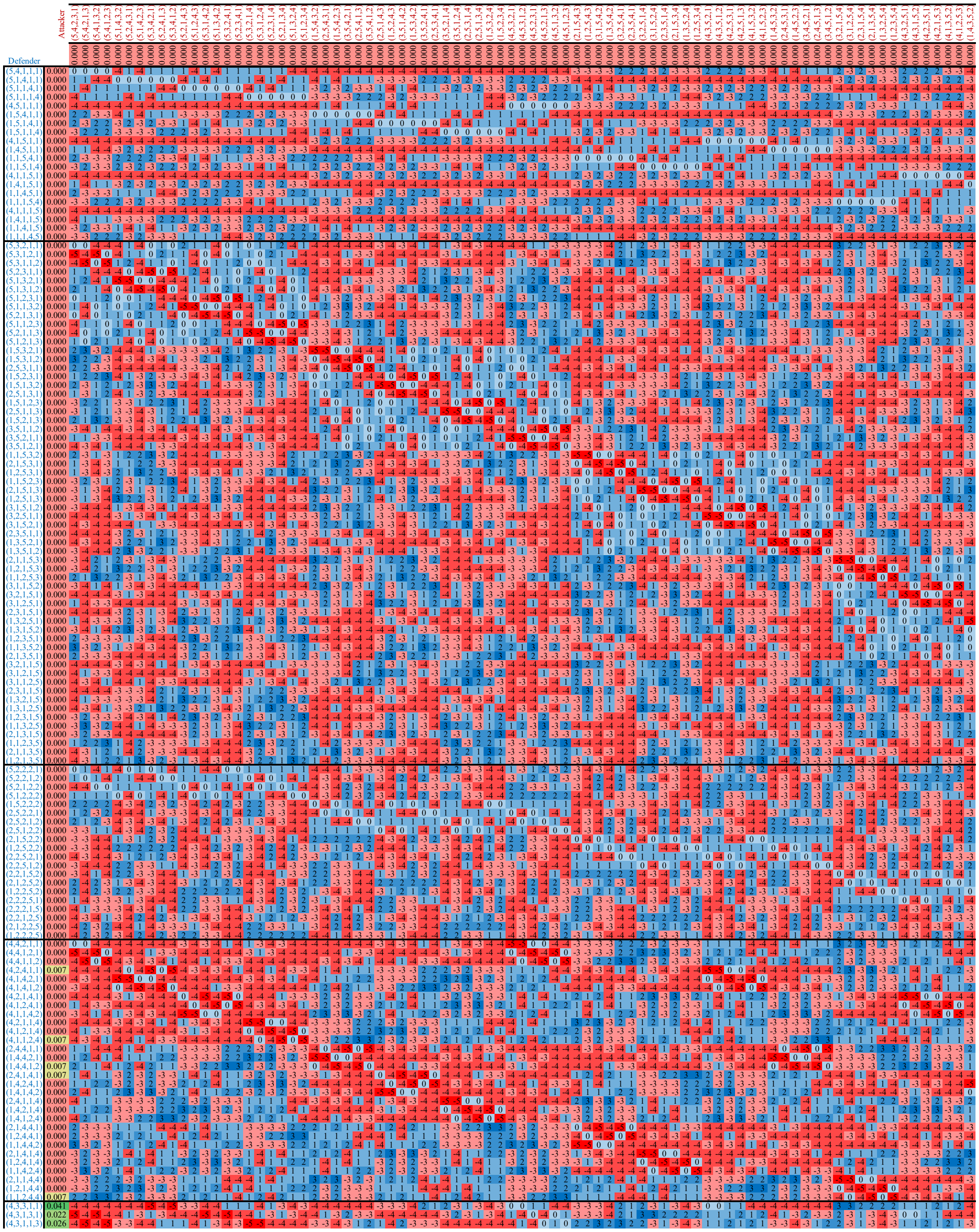
Section 2 of 16



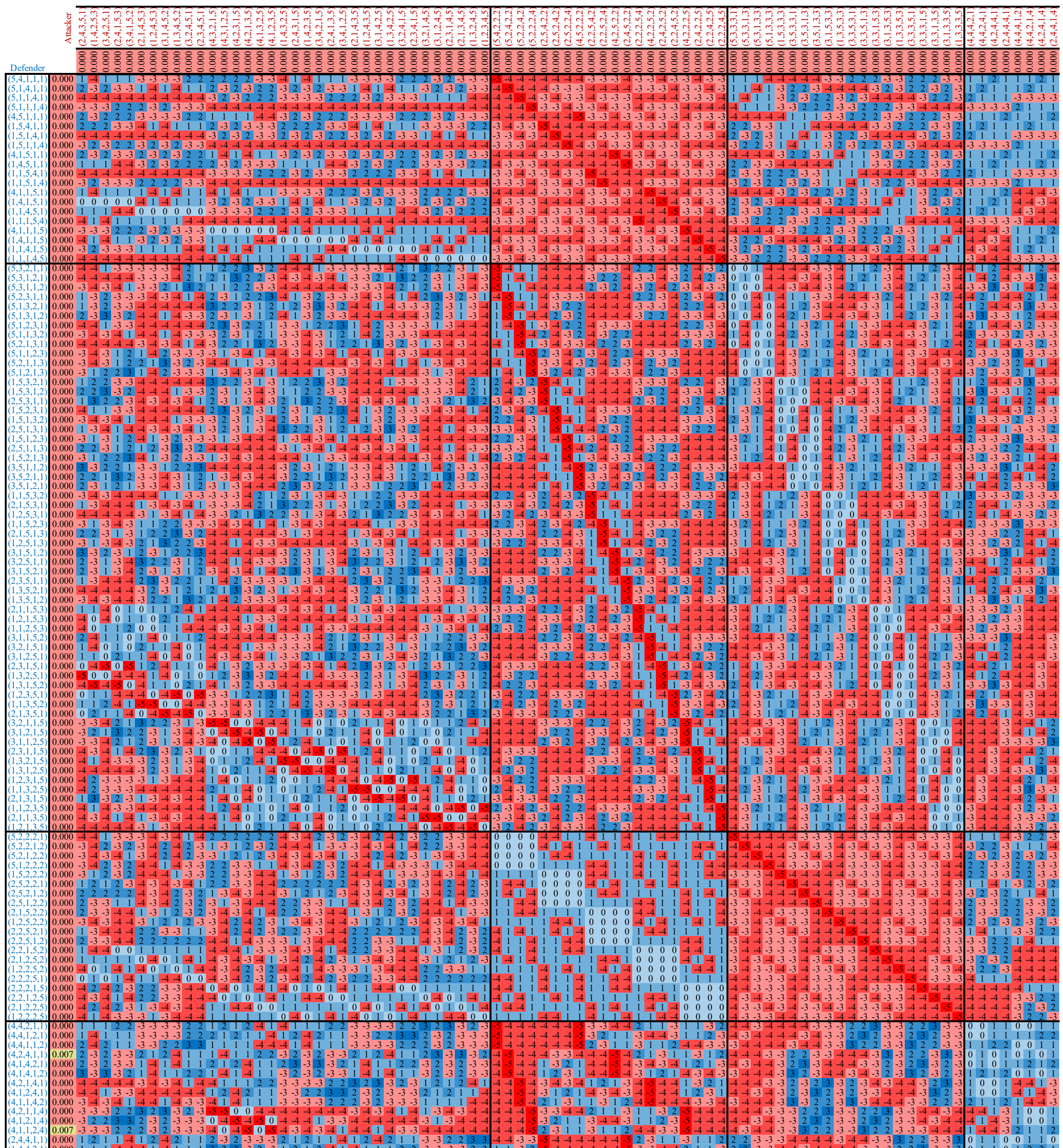
Section 3 of 16



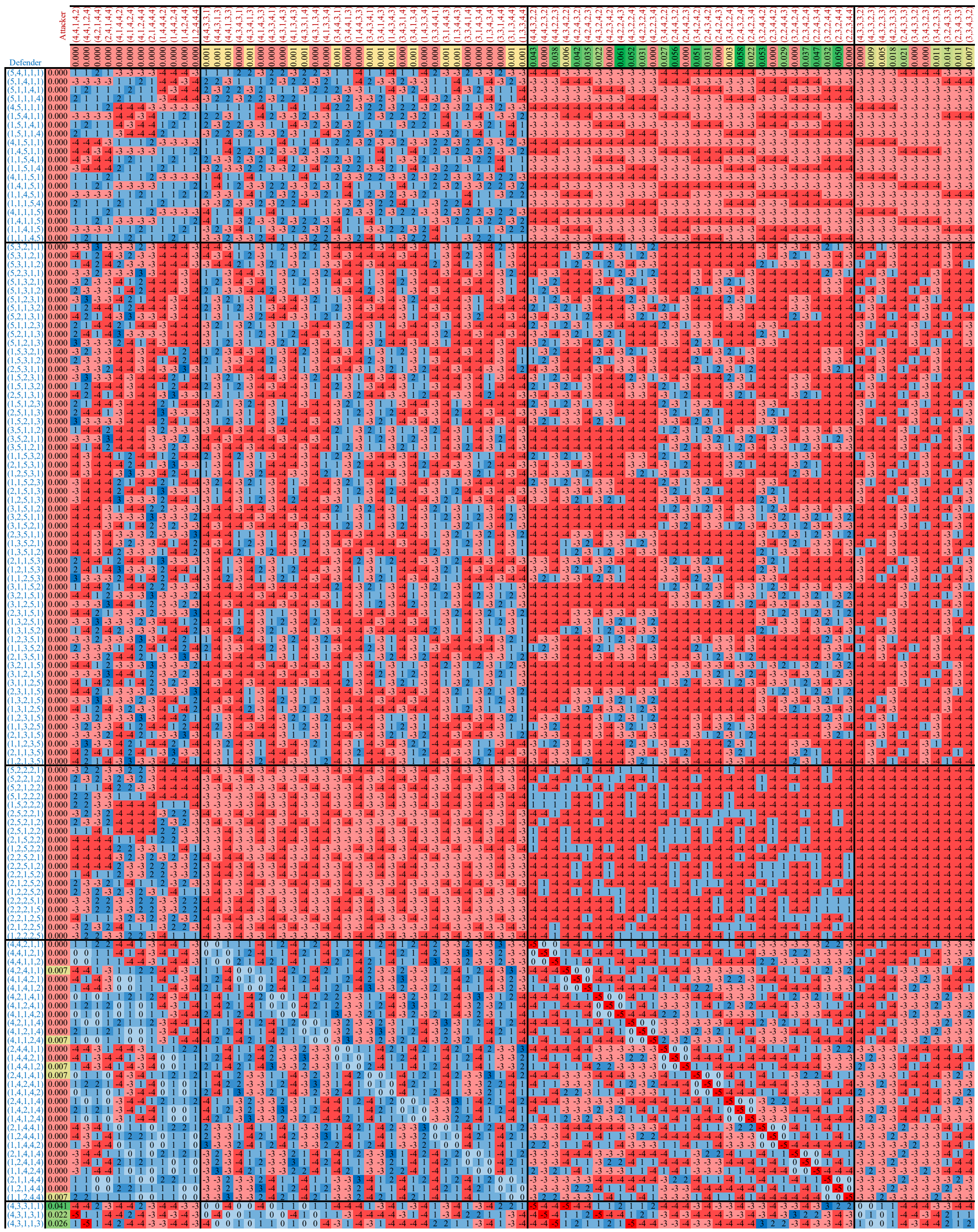
Section 4 of 16



Section 5 of 16



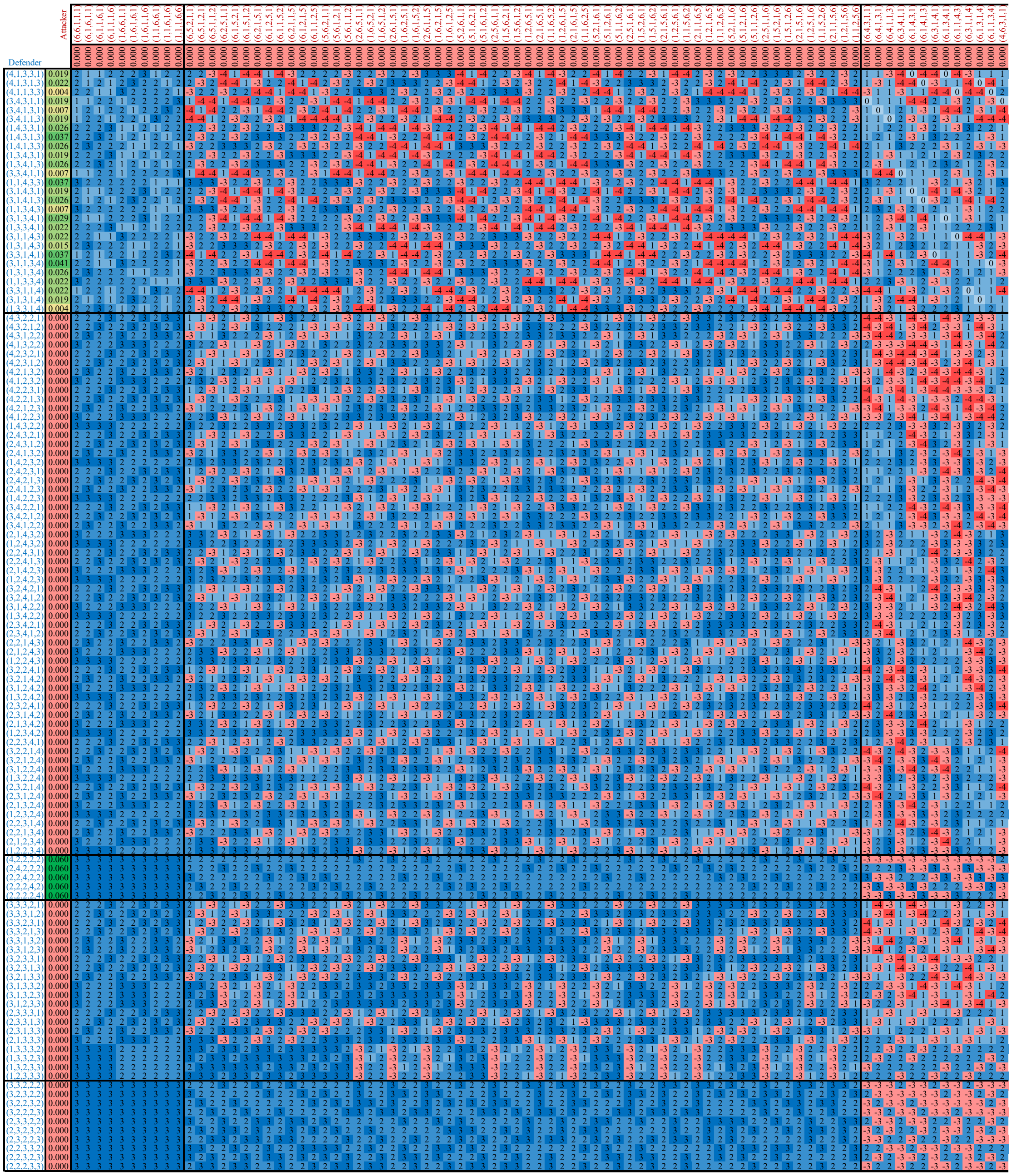
Section 6 of 16



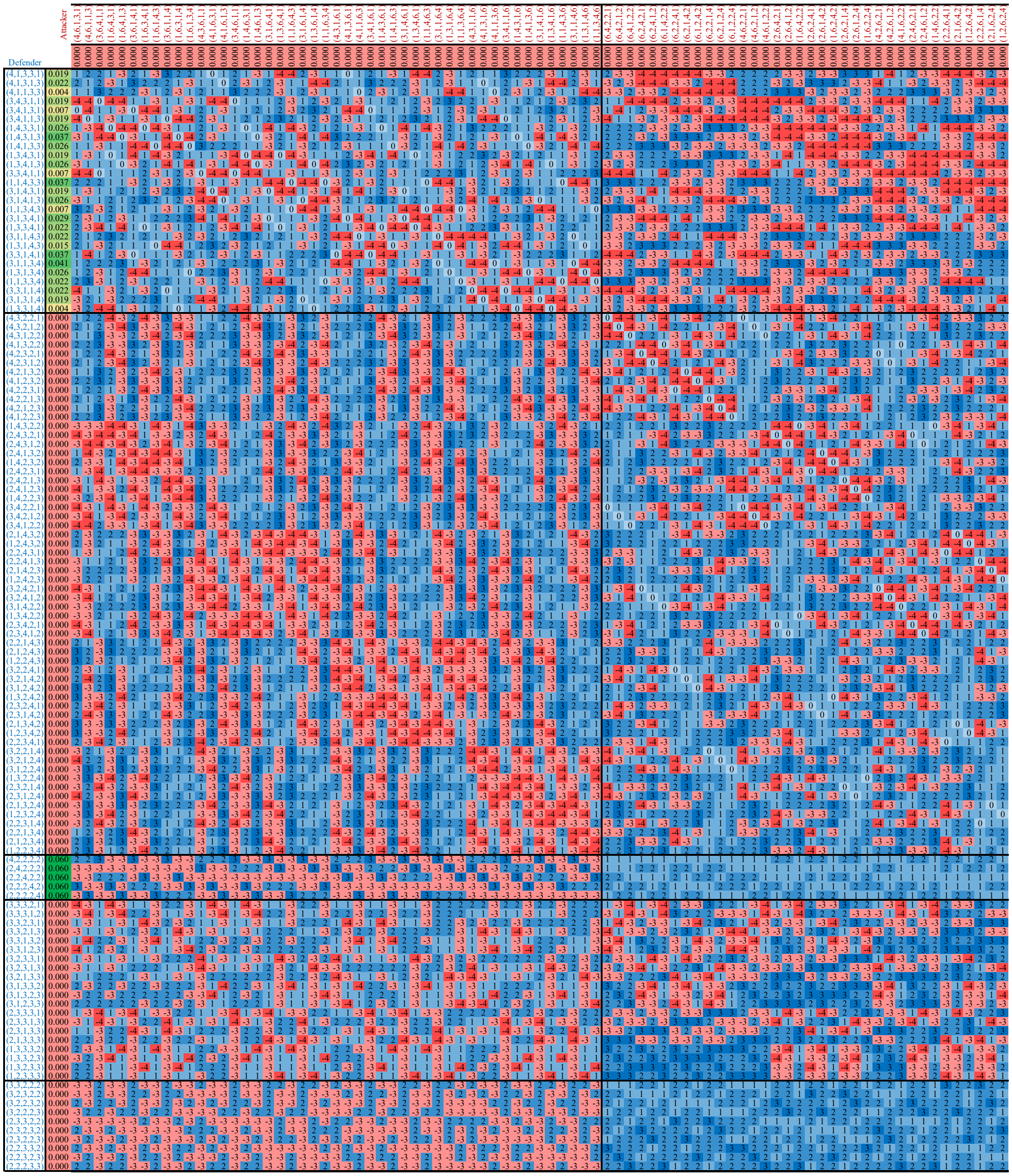
Section 7 of 16

Defender		Attacker (3,3,4,2,3)	(3,3,2,4,3)	(3,2,3,4,3)	(2,3,3,4,3)	(3,3,3,4,2)	(3,3,3,2,4)	(3,3,2,3,4)	(3,2,3,3,4)	(2,3,3,3,4)	(3,3,3,3,3)
		0.000	0.007	0.000	0.000	0.025	0.023	0.009	0.000	0.000	0.000
(5,4,1,1,1)	0.000	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
(5,1,4,1,1)	0.000	-4	-3	-3	-3	-3	-3	-3	-3	-3	-3
(5,1,1,4,1)	0.000	-3	-4	-4	-4	-4	-3	-3	-3	-3	-3
(5,1,1,1,4)	0.000	-3	-3	-3	-3	-3	-4	-4	-4	-4	-3
(4,5,1,1,1)	0.000	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
(1,5,4,1,1)	0.000	-4	-3	-3	-3	-3	-3	-3	-3	-3	-3
(1,5,1,4,1)	0.000	-3	-4	-4	-4	-4	-3	-3	-3	-3	-3
(1,5,1,1,4)	0.000	-3	-3	-3	-3	-3	-4	-4	-4	-4	-3
(4,1,5,1,1)	0.000	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
(1,4,5,1,1)	0.000	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
(1,1,5,4,1)	0.000	-3	-4	-4	-4	-4	-3	-3	-3	-3	-3
(1,1,5,1,4)	0.000	-3	-3	-3	-3	-3	-4	-4	-4	-4	-3
(4,1,1,5,1)	0.000	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
(1,4,1,5,1)	0.000	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
(1,1,4,5,1)	0.000	-4	-3	-3	-3	-3	-3	-3	-3	-3	-3
(1,1,1,5,4)	0.000	-3	-3	-3	-3	-3	-4	-4	-4	-4	-3
(4,1,1,1,5)	0.000	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
(1,4,1,1,5)	0.000	-3	-3	-3	-3	-3	-3	-3	-3	-3	-3
(1,1,4,1,5)	0.000	-4	-3	-3	-3	-3	-3	-3	-3	-3	-3
(1,1,1,4,5)	0.000	-3	-4	-4	-4	-4	-3	-3	-3	-3	-3
(5,3,2,1,1)	0.000	-4	1	-3	-4	-4	-4	1	-3	-4	-4
(5,3,1,2,1)	0.000	1	-4	-3	-4	-4	1	-4	-3	-4	-4
(5,3,1,1,2)	0.000	-4	-4	-3	-4	1	-4	-4	-3	-4	-4
(5,2,3,1,1)	0.000	-4	-3	1	-4	-4	-4	-3	1	-4	-4
(5,1,3,2,1)	0.000	-4	-3	-4	-4	-4	1	-3	-4	-4	-4
(5,1,3,1,2)	0.000	-4	-3	-4	-4	1	-4	-3	-4	-4	-4
(5,1,2,3,1)	0.000	-3	-4	-4	-4	-4	-3	1	-4	-4	-4
(5,1,1,3,2)	0.000	-3	-4	-4	-4	-4	-3	-4	-4	-4	-4
(5,2,1,3,1)	0.000	-3	-4	-4	-4	-4	-3	-4	1	-4	-4
(5,1,1,2,3)	0.000	1	-4	-4	-4	-3	-4	-4	-4	-4	-4
(5,2,1,1,3)	0.000	-4	-4	1	-4	-3	-4	-4	-4	-4	-4
(5,1,2,1,3)	0.000	-4	1	-4	-4	-3	-4	-4	-4	-4	-4
(1,5,3,2,1)	0.000	-4	-3	-4	-4	-4	1	-3	-4	-4	-4
(1,5,3,1,2)	0.000	-4	-3	-4	-4	1	-4	-3	-4	-4	-4
(2,5,3,1,1)	0.000	-4	-3	-4	1	-4	-4	-3	-4	1	-4
(1,5,2,3,1)	0.000	-3	-4	-4	-4	-4	-3	1	-4	-4	-4
(1,5,1,3,2)	0.000	-3	-4	-4	-4	-4	-3	-4	-4	-4	-4
(2,5,1,3,1)	0.000	-3	-4	-4	-4	-4	-3	-4	-4	1	-4
(1,5,1,2,3)	0.000	1	-4	-4	-4	-3	-4	-4	-4	-4	-4
(2,5,1,1,3)	0.000	-4	-4	-4	1	-3	-4	-4	-4	-4	-4
(1,5,2,1,3)	0.000	-4	1	-4	-4	-3	-4	-4	-4	-4	-4
(3,5,1,1,2)	0.000	-4	-4	-4	-3	1	-4	-4	-4	-3	-4
(3,5,2,1,1)	0.000	-4	1	-4	-3	-4	-4	1	-4	-3	-4
(3,5,1,2,1)	0.000	1	-4	-4	-3	-4	1	-4	-4	-3	-4
(1,1,5,3,2)	0.000	-3	-4	-4	-4	-4	-3	-4	-4	-4	-4
(2,1,5,3,1)	0.000	-3	-4	-4	-4	-4	-3	-4	-4	1	-4
(1,2,5,3,1)	0.000	-3	-4	-4	-4	-4	-3	-4	1	-4	-4
(1,1,5,2,3)	0.000	1	-4	-4	-4	-3	-4	-4	-4	-4	-4
(2,1,5,1,3)	0.000	-4	-4	-4	1	-3	-4	-4	-4	-4	-4
(1,2,5,1,3)	0.000	-4	-4	1	-4	-3	-4	-4	-4	-4	-4
(3,1,5,1,2)	0.000	-4	-4	-4	-3	1	-4	-4	-4	-3	-4
(3,2,5,1,1)	0.000	-4	-4	1	-3	-4	-4	-4	1	-3	-4
(3,1,5,2,1)	0.000	1	-4	-4	-3	-4	1	-4	-4	-3	-4
(2,3,5,1,1)	0.000	-4	-4	-3	1	-4	-4	-4	-3	1	-4
(1,3,5,2,1)	0.000	1	-4	-3	-4	-4	1	-4	-3	-4	-4
(1,3,5,1,2)	0.000	-4	-4	-3	-4	1	-4	-4	-3	-4	-4
(2,1,1,5,3)	0.000	-4	-4	-4	1	-3	-4	-4	-4	-4	-4
(1,2,1,5,3)	0.000	-4	-4	1	-4	-3	-4	-4	-4	-4	-4
(1,1,2,5,3)	0.000	-4	1	-4	-4	-3	-4	-4	-4	-4	-4
(3,1,1,5,2)	0.000	-4	-4	-4	-3	1	-4	-4	-4	-3	-4
(3,2,1,5,1)	0.000	-4	-4	1	-3	-4	-4	-4	1	-3	-4
(3,1,2,5,1)	0.000	-4	1	-4	-3	-4	-4	1	-4	-3	-4
(2,3,1,5,1)	0.000	-4	-4	-3	1	-4	-4	-4	-3	1	-4
(1,3,2,5,1)	0.000	-4	1	-3	-4	-4	-4	1	-3	-4	-4
(1,3,1,5,2)	0.000	-4	-4	-3	-4	1	-4	-4	-3	-4	-4
(1,2,3,5,1)	0.000	-4	-3	1	-4	-4	-4	-3	1	-4	-4
(1,1,3,5,2)	0.000	-4	-3	-4	-4	1	-4	-3	-4	-4	-4
(2,1,3,5,1)	0.000	-4	-3	-4	1	-4	-4	-3	-4	1	-4
(3,2,1,1,5)	0.000	-4	-4	1	-3	-4	-4	-4	1	-3	-4
(3,1,2,1,5)	0.000	-4	1	-4	-3	-4	-4	1	-4	-3	-4
(3,1,1,2,5)	0.000	1	-4	-4	-3	-4	1	-4	-4	-3	-4
(2,3,1,1,5)	0.000	-4	-4	-3	1	-4	-4	-4	-3	1	-4
(1,3,2,1,5)	0.000	-4	1	-3	-4	-4	-4	1	-3	-4	-4
(1,3,1,2,5)	0.000	1	-4	-3	-4	-4	1	-4	-3	-4	-4
(1,2,3,1,5)	0.000	-4	-3	1	-4	-4	-4	-3	1	-4	-4
(1,1,3,2,5)	0.000	-4	-3	-4	-4	-4	1	-3	-4	-4	-4
(2,1,3,1,5)	0.000	-4	-3	-4	1	-4	-4	-3	-4	1	-4
(1,1,2,3,5)	0.000	-3	-4	-4	-4	-4	-3	1	-4	-4	-4
(2,1,1,3,5)	0.000	-3	-4	-4	-4	-4	-3	-4	-4	1	-4
(1,2,1,3,5)	0.000	-3	-4	-4	-4	-4	-3	-4	1	-4	-4
(5,2,2,2,1)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(5,2,2,1,2)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(5,2,1,2,2)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(5,1,2,2,2)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(1,5,2,2,2)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,5,2,2,1)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,5,2,1,2)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,5,1,2,2)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,1,5,2,2)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(1,2,5,2,2)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,2,5,2,1)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,2,5,1,2)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,2,1,5,2)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,1,2,5,2)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(1,2,2,5,2)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,2,2,5,1)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,2,2,1,5)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,2,1,2,5)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,1,2,2,5)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(1,2,2,2,5)	0.000	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(4,4,2,1,1)	0.000	-3	2	-3	-3	-3	-3	2	-3	-3	-3
(4,4,1,2,1)	0.000	2	-3	-3	-3	-3	2	-3	-3	-3	-3
(4,4,1,1,2)	0.000	-3	-3	-3	-3	2	-3	-3	-3	-3	-3
(4,2,4,1,1)	0.007	-4	-3	2	-3	-3	-3	-3	2	-3	-3
(4,1,4,2,1)	0.000	1	-3	-3	-3	-3	2	-3	-3	-3	-3
(4,1,4,1,2)	0.000	-4	-3	-3	-3	2	-3	-3	-3	-3	-3
(4,2,1,4,1)	0.000	-3	-4	1	-4	-4	-3	-3	2	-3	-3
(4,1,2,4,1)	0.000	-3	1	-4	-4	-4	-3	2	-3	-3	-3
(4,1,1,4,2)	0.000	-3	-4	-4	-4	1	-3	-3	-3	-3	-3
(4,2,1,1,4)	0.000	-3	-3	2	-3	-3	-4	-4	1	-4	-3
(4,1,2,1,4)	0.000	-3	2	-3	-3	-3	-4	1	-4	-4	-3
(4,1,1,2,4)	0.007	2	-3	-3	-3	-3	1	-4	-4	-4	-3
(2,4,4,1,1)	0.000	-4	-3	-3	2	-3	-3	-3	-3	2	-3
(1,4,4,2,1)	0.000	1	-3	-3	-3	-3	2	-3	-3	-3	-3
(1,4,4,1,2)	0.007	-4	-3	-3	-3	2	-3	-3	-3	-3	-3
(2,4,1,4,1)	0.007	-3	-4	-4	1	-4	-3	-3	-3	2	-3
(1,4,2,4,1)	0.000	-3	1	-4	-4	-4	-3	2	-3	-3	-3
(1,4,1,4,2)	0.000	-3	-4	-4	-4	1	-3	-3	-3	-3	-3
(2,4,1,1,4)	0.000	-3	-3	-3	2	-3	-4	-4	-4	1	-3
(1,4,2,1,4)	0.000	-3	2	-3	-3	-3	-4	1	-4	-4	-3
(1,4,1,2,4)	0.000	2	-3	-3	-3	-3	1	-4	-4	-4	-3
(2,1,4,4,1)	0.000	-4	-4	-4	1	-4	-3	-3	-3	2	-3
(1,2,4,4,1)	0.000	-4	-4	1	-4	-4	-3	-3	2	-3	-3
(1,1,4,4,2)	0.000	-4	-4	-4	-4	1	-3	-3	-3	-3	-3
(2,1,4,1,4)	0.000	-4	-3	-3	2	-3	-4	-4	-4	1	-3
(1,2,4,1,4)	0.000	-4	-3	2	-3	-3	-4	-4	1	-4	-3
(1,1,4,2,4)	0.000	1	-3	-3	-3	-3	1	-4	-4	-4	-3
(2,1,1,4,4)	0.000	-3	-4	-4	1	-4	-4	-4	-4	1	-3
(1,2,1,4,4)	0.000	-3	-4	1	-4	-4	-4	-4	1	-4	-3
(1,1,2,4,4)	0.007	-3	1	-4	-4	-4	-4	1	-4	-4	-3
(4,3,3,1,1)	0.041	-4	2	2	1	1	1	2	2	1	1
(4,3,1,3,1)	0.022	2	-4	-3	-4	-4	2	1	2	1	1
(4,3,1,1,3)	0.026	1	1	2	1	2	-4	-4	-3	-4	1

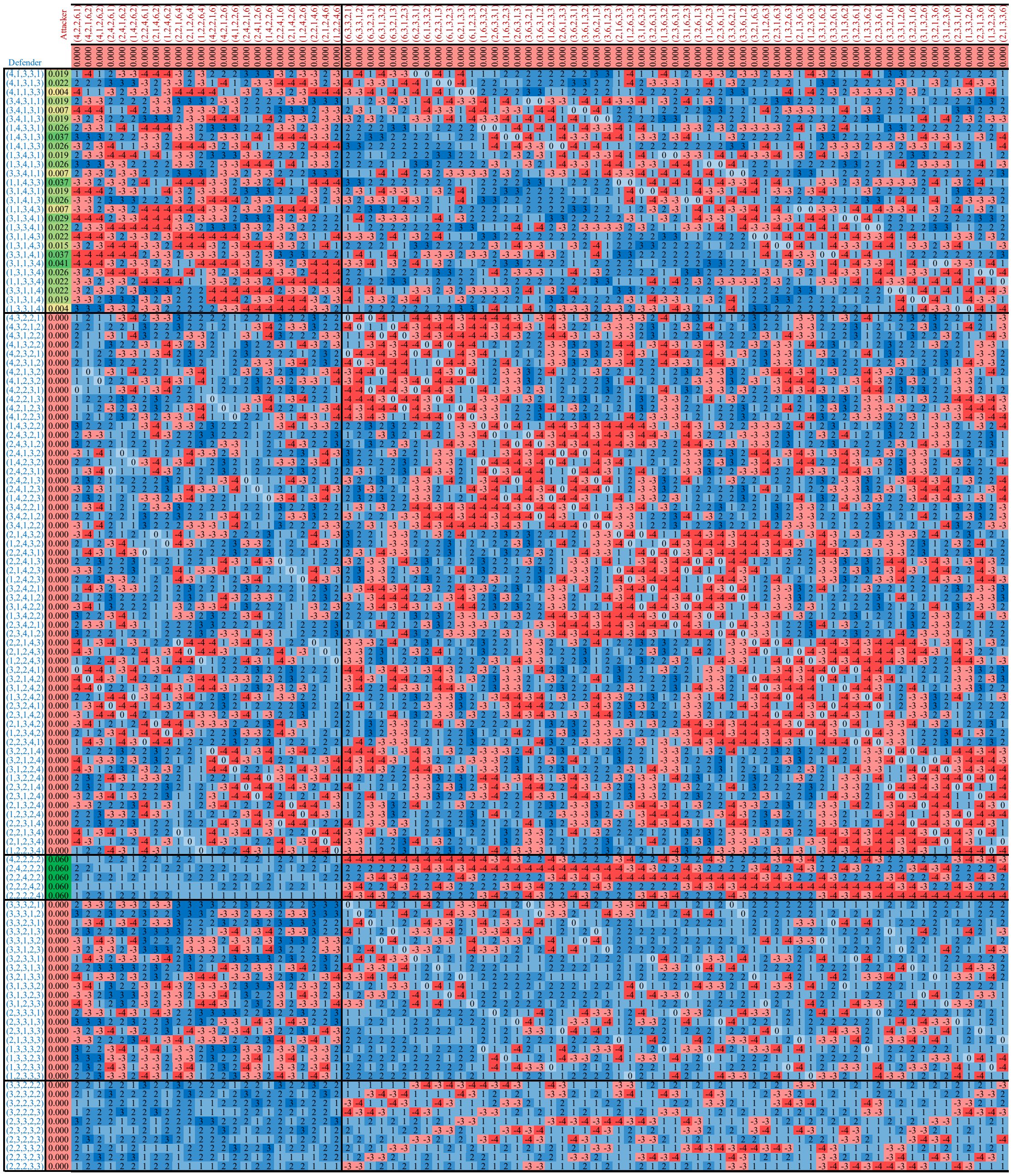
Section 8 of 16



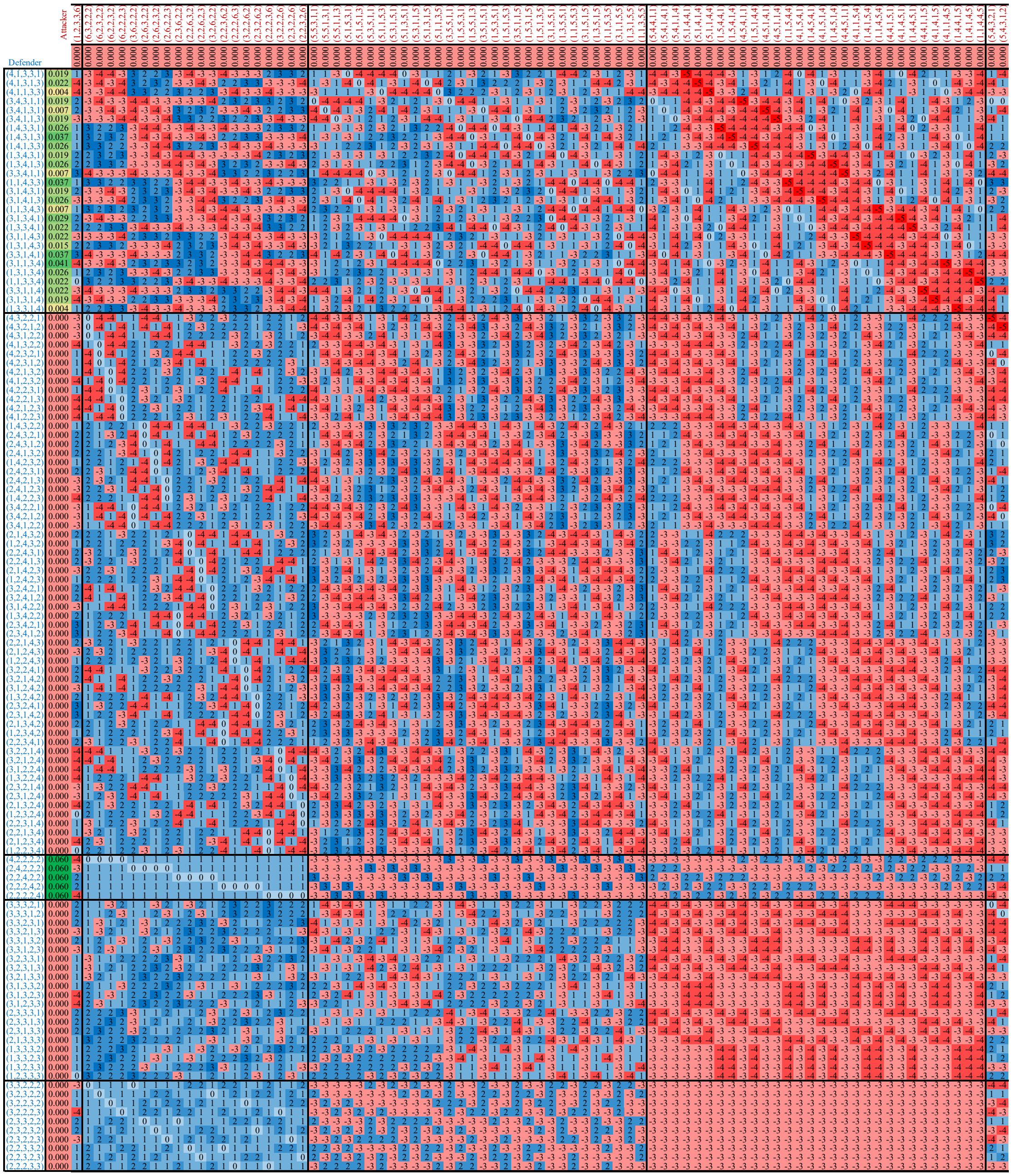
Section 9 of 16



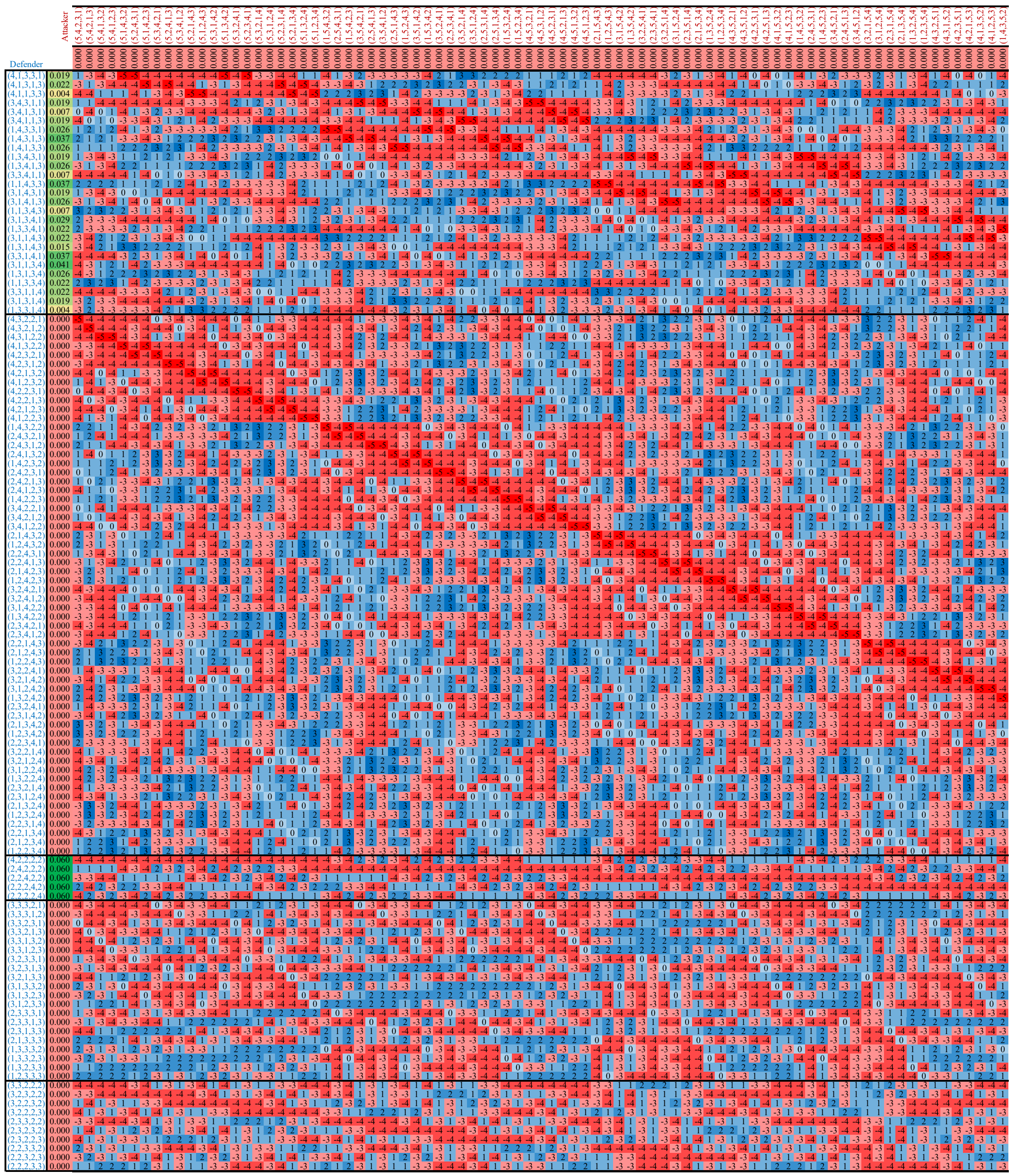
Section 10 of 16



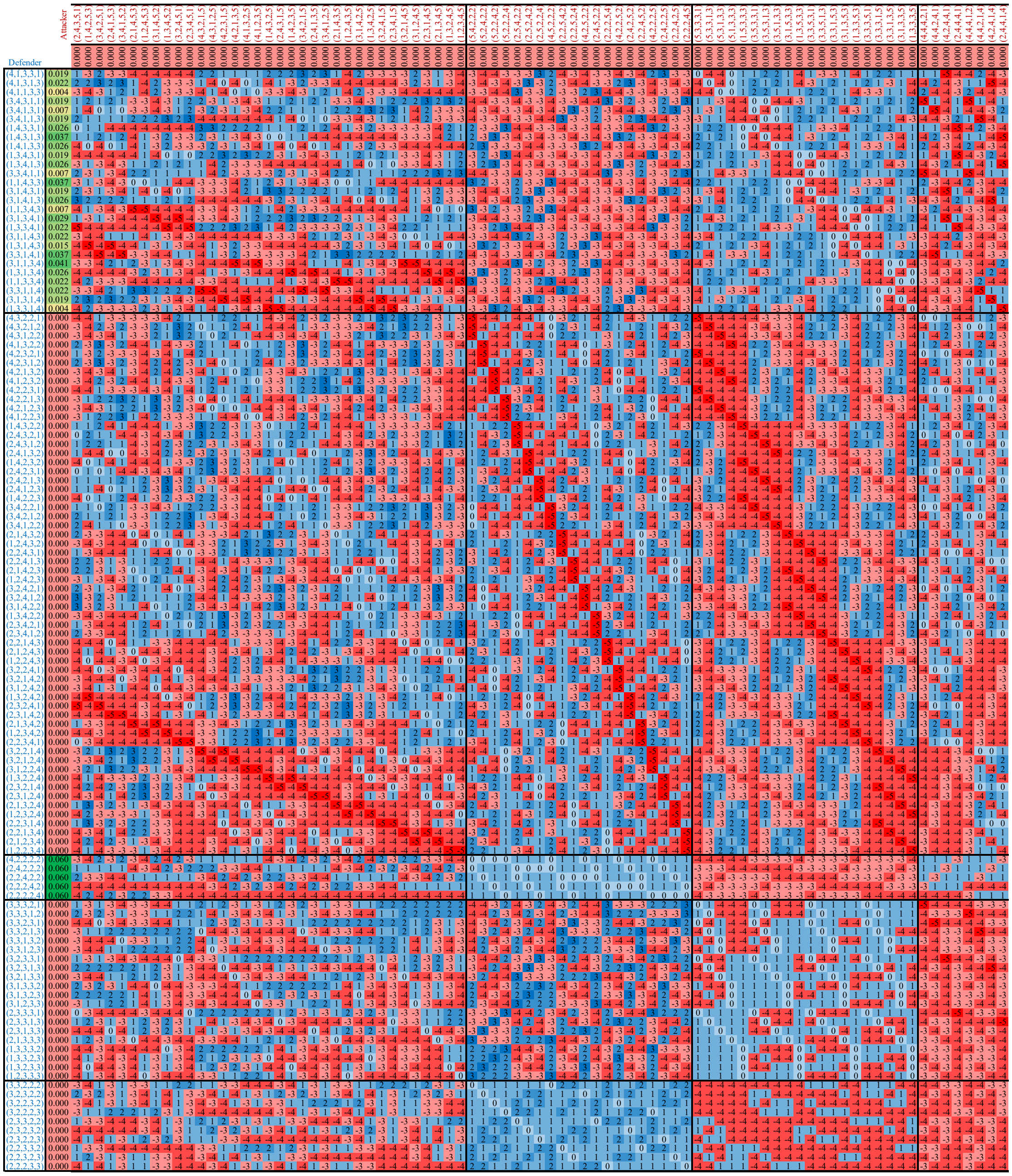
Section 11 of 16



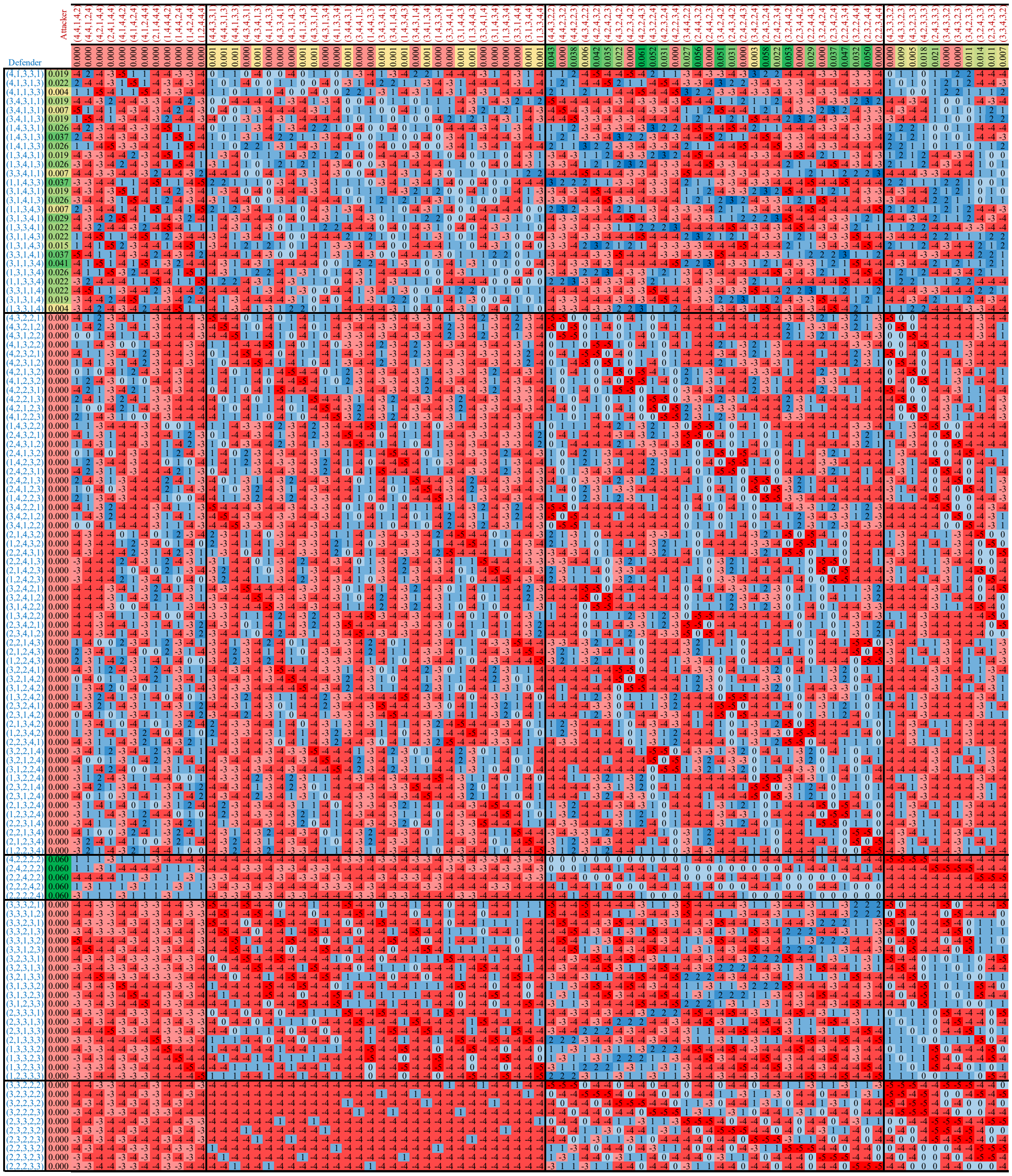
Section 12 of 16



Section 13 of 16



Section 14 of 16



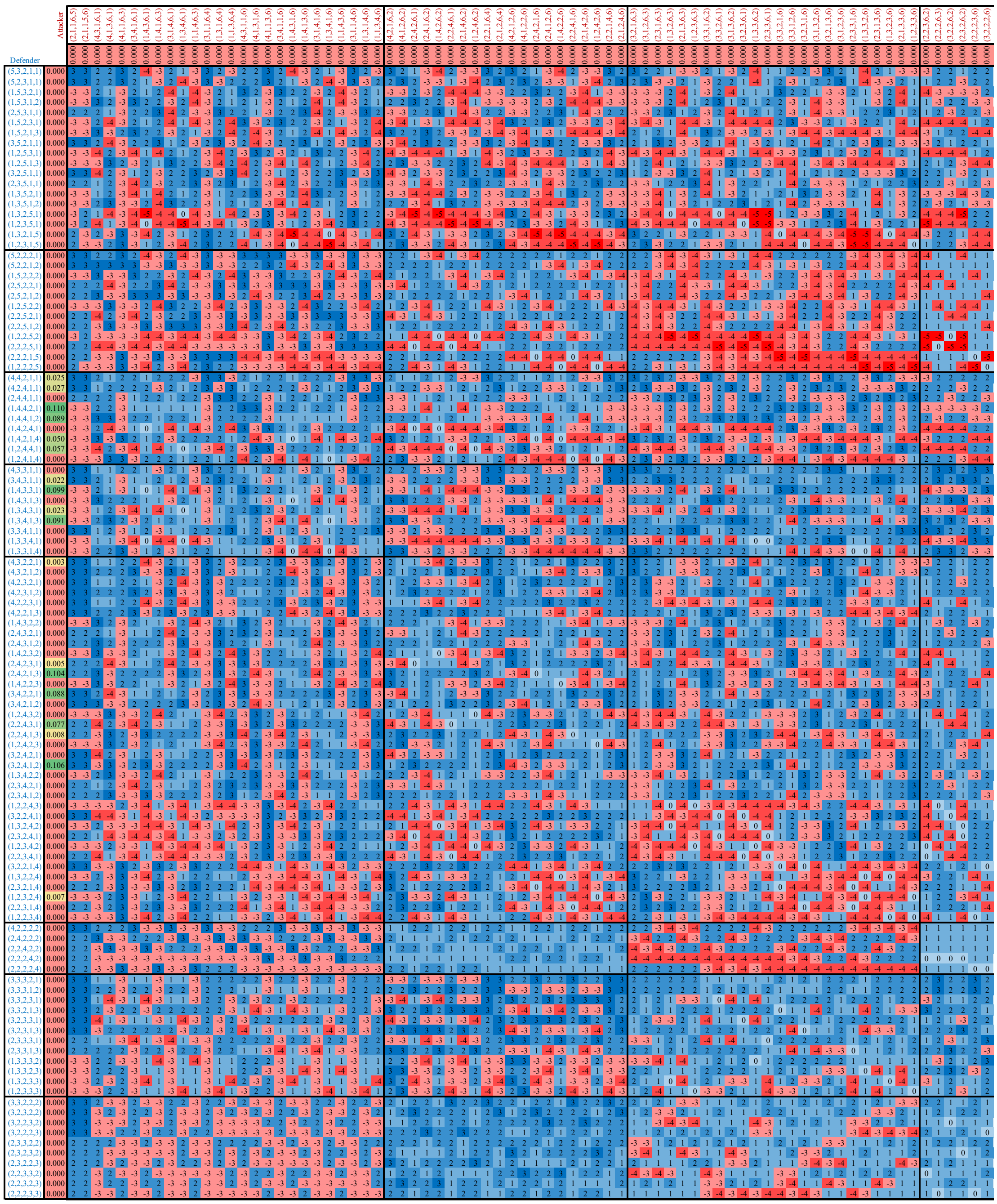
Section 15 of 16

Defender \ Attacker		(3,3,4,2,3)	(3,3,2,4,3)	(3,2,3,4,3)	(2,3,3,4,3)	(3,3,3,4,2)	(3,3,3,2,4)	(3,3,2,3,4)	(3,2,3,3,4)	(2,3,3,3,4)	(3,3,3,3,3)
		0.000	0.007	0.000	0.000	0.025	0.023	0.009	0.000	0.000	0.000
(4,1,3,3,1)	0.019	-3	-3	-4	-4	-4	2	2	1	1	1
(4,1,3,1,3)	0.022	-4	2	1	1	2	-4	-3	-4	-4	1
(4,1,1,3,3)	0.004	2	-4	-4	-4	-3	-3	-4	-4	-4	1
(3,4,3,1,1)	0.019	-4	2	1	2	1	1	2	1	2	1
(3,4,1,3,1)	0.007	2	-4	-4	-3	-4	2	1	1	2	1
(3,4,1,1,3)	0.019	1	1	1	2	2	-4	-4	-4	-3	1
(1,4,3,3,1)	0.026	-3	-3	-4	-4	-4	2	2	1	1	1
(1,4,3,1,3)	0.037	-4	2	1	1	2	-4	-3	-4	-4	1
(1,4,1,3,3)	0.026	2	-4	-4	-4	-3	-3	-4	-4	-4	1
(1,3,4,3,1)	0.019	1	-4	-3	-4	-4	2	1	2	1	1
(1,3,4,1,3)	0.026	0	1	2	1	2	-4	-4	-3	-4	1
(3,3,4,1,1)	0.007	0	1	2	2	1	1	1	2	2	1
(1,1,4,3,3)	0.037	1	-4	-4	-4	-3	-3	-4	-4	-4	1
(3,1,4,3,1)	0.019	1	-4	-4	-3	-4	2	1	1	2	1
(3,1,4,1,3)	0.026	0	1	1	2	2	-4	-4	-4	-3	1
(1,1,3,4,3)	0.007	-4	1	0	0	1	-4	-3	-4	-4	1
(3,1,3,4,1)	0.029	-4	1	0	1	0	1	2	1	2	1
(1,3,3,4,1)	0.022	-4	1	1	0	0	1	2	2	1	1
(3,1,1,4,3)	0.022	1	0	0	1	1	-4	-4	-4	-3	1
(1,3,1,4,3)	0.015	1	0	1	0	1	-4	-4	-3	-4	1
(3,3,1,4,1)	0.037	1	0	1	1	0	1	1	2	2	1
(3,1,1,3,4)	0.041	2	-4	-4	-3	-4	1	0	0	1	1
(1,3,1,3,4)	0.026	2	-4	-3	-4	-4	1	0	1	0	1
(1,1,3,3,4)	0.022	-3	-3	-4	-4	-4	1	1	0	0	1
(3,3,1,1,4)	0.022	1	1	2	2	1	0	0	1	1	1
(3,1,3,1,4)	0.019	-4	2	1	2	1	0	1	0	1	1
(1,3,3,1,4)	0.004	-4	2	2	1	1	0	1	1	0	1
(4,3,2,2,1)	0.000	1	1	-3	-4	-4	1	1	-3	-4	-4
(4,3,2,1,2)	0.000	-4	1	-3	-4	1	-4	1	-3	-4	-4
(4,3,1,2,2)	0.000	1	-4	-3	-4	1	1	-4	-3	-4	-4
(4,1,3,2,2)	0.000	-4	-3	-4	-4	1	1	-3	-4	-4	-4
(4,2,3,2,1)	0.000	-4	-3	1	-4	-4	1	-3	1	-4	-4
(4,2,3,1,2)	0.000	-4	-3	1	-4	1	-4	-3	1	-4	-4
(4,2,1,3,2)	0.000	-3	-4	-4	-4	-4	-3	-4	1	-4	-4
(4,1,2,3,2)	0.000	-3	-4	-4	-4	-4	-3	1	-4	-4	-4
(4,2,2,3,1)	0.000	-3	-4	-4	-4	-4	-3	1	1	-4	-4
(4,2,2,1,3)	0.000	-4	1	1	-4	-3	-4	-4	-4	-4	-4
(4,2,1,2,3)	0.000	1	-4	1	-4	-3	-4	-4	-4	-4	-4
(4,1,2,2,3)	0.000	1	1	-4	-4	-3	-4	-4	-4	-4	-4
(1,4,3,2,2)	0.000	-4	-3	-4	-4	1	1	-3	-4	-4	-4
(2,4,3,2,1)	0.000	-4	-3	-4	1	-4	1	-3	-4	1	-4
(2,4,3,1,2)	0.000	-4	-3	-4	1	1	-4	-3	-4	1	-4
(2,4,1,3,2)	0.000	-3	-4	-4	-4	-4	-3	-4	-4	1	-4
(1,4,2,3,2)	0.000	-3	-4	-4	-4	-4	-3	1	-4	-4	-4
(2,4,2,3,1)	0.000	-3	-4	-4	-4	-4	-3	1	-4	1	-4
(2,4,2,1,3)	0.000	-4	1	-4	1	-3	-4	-4	-4	-4	-4
(2,4,1,2,3)	0.000	1	-4	-4	1	-3	-4	-4	-4	-4	-4
(1,4,2,2,3)	0.000	1	1	-4	-4	-3	-4	-4	-4	-4	-4
(3,4,2,2,1)	0.000	1	1	-4	-3	-4	1	1	-4	-3	-4
(3,4,2,1,2)	0.000	-4	1	-4	-3	1	-4	1	-4	-3	-4
(3,4,1,2,2)	0.000	1	-4	-4	-3	1	1	-4	-4	-3	-4
(2,1,4,3,2)	0.000	-4	-4	-4	-4	-4	-3	-4	-4	1	-4
(1,2,4,3,2)	0.000	-4	-4	-4	-4	-4	-3	-4	1	-4	-4
(2,2,4,3,1)	0.000	-4	-4	-4	-4	-4	-3	-4	1	1	-4
(2,2,4,1,3)	0.000	-5	-4	1	1	-3	-4	-4	-4	-4	-4
(2,1,4,2,3)	0.000	0	-4	-4	1	-3	-4	-4	-4	-4	-4
(1,2,4,2,3)	0.000	0	-4	1	-4	-3	-4	-4	-4	-4	-4
(3,2,4,2,1)	0.000	0	-4	1	-3	-4	1	-4	1	-3	-4
(3,2,4,1,2)	0.000	-5	-4	1	-3	1	-4	-4	1	-3	-4
(3,1,4,2,2)	0.000	0	-4	-4	-3	1	1	-4	-4	-3	-4
(1,3,4,2,2)	0.000	0	-4	-3	-4	1	1	-4	-3	-4	-4
(2,3,4,2,1)	0.000	0	-4	-3	1	-4	1	-4	-3	1	-4
(2,3,4,1,2)	0.000	-5	-4	-3	1	1	-4	-4	-3	1	-4
(2,2,1,4,3)	0.000	-4	-5	0	0	-4	-4	-4	-4	-4	-4
(2,1,2,4,3)	0.000	-4	0	-5	0	-4	-4	-4	-4	-4	-4
(1,2,2,4,3)	0.000	-4	0	0	-5	-4	-4	-4	-4	-4	-4
(3,2,2,4,1)	0.000	-4	0	0	-4	-5	-4	1	1	-3	-4
(3,2,1,4,2)	0.000	-4	-5	0	-4	0	-4	-4	1	-3	-4
(3,1,2,4,2)	0.000	-4	0	-5	-4	0	-4	1	-4	-3	-4
(1,3,2,4,2)	0.000	-4	0	-4	-5	0	-4	1	-3	-4	-4
(2,3,2,4,1)	0.000	-4	0	-4	0	-5	-4	1	-3	1	-4
(2,3,1,4,2)	0.000	-4	-5	-4	0	0	-4	-4	-3	1	-4
(2,1,3,4,2)	0.000	-4	-4	-5	0	0	-4	-3	-4	1	-4
(1,2,3,4,2)	0.000	-4	-4	0	-5	0	-4	-3	1	-4	-4
(2,2,3,4,1)	0.000	-4	-4	0	0	-5	-4	-3	1	1	-4
(3,2,2,1,4)	0.000	-4	1	1	-3	-4	-5	0	0	-4	-4
(3,2,1,2,4)	0.000	1	-4	1	-3	-4	0	-5	0	-4	-4
(3,1,2,2,4)	0.000	1	1	-4	-3	-4	0	0	-5	-4	-4
(1,3,2,2,4)	0.000	1	1	-3	-4	-4	0	0	-4	-5	-4
(2,3,2,1,4)	0.000	-4	1	-3	1	-4	-5	0	-4	0	-4
(2,3,1,2,4)	0.000	1	-4	-3	1	-4	0	-5	-4	0	-4
(2,1,3,2,4)	0.000	-4	-3	-4	1	-4	0	-4	-5	0	-4
(1,2,3,2,4)	0.000	-4	-3	1	-4	-4	0	-4	0	-5	-4
(2,2,3,1,4)	0.000	-4	-3	1	1	-4	-5	-4	0	0	-4
(2,2,1,3,4)	0.000	-3	-4	-4	-4	-4	-5	0	0	0	-4
(2,1,2,3,4)	0.000	-3	-4	-4	-4	-4	-4	0	-5	0	-4
(1,2,2,3,4)	0.000	-3	-4	-4	-4	-4	-4	0	0	-5	-4
(4,2,2,2,2)	0.060	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,4,2,2,2)	0.060	-4	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,2,4,2,2)	0.060	-5	-4	-4	-4	-4	-4	-4	-4	-4	-4
(2,2,2,4,2)	0.060	-4	-5	-5	-5	-5	-4	-4	-4	-4	-4
(2,2,2,2,4)	0.060	-4	-4	-4	-4	-4	-5	-5	-5	-5	-4
(3,3,3,2,1)	0.000	0	1	1	1	0	0	1	1	1	0
(3,3,3,1,2)	0.000	-5	1	1	1	0	0	1	1	1	0
(3,3,2,3,1)	0.000	1	0	-4	-4	-5	1	0	1	0	0
(3,3,2,1,3)	0.000	0	0	1	1	1	-5	0	-4	-4	0
(3,3,1,3,2)	0.000	1	-5	-4	-4	0	1	0	1	1	0
(3,3,1,2,3)	0.000	0	0	1	1	1	0	-5	-4	-4	0
(3,2,3,3,1)	0.000	-4	-4	0	-4	-5	1	1	0	1	0
(3,2,3,1,3)	0.000	-5	1	0	1	1	-5	-4	0	-4	0
(3,2,1,3,3)	0.000	1	-5	0	-4	-4	-4	-5	0	-4	0
(3,1,3,3,2)	0.000	-4	-4	-5	-4	0	1	1	0	1	0
(3,1,3,2,3)	0.000	0	1	0	1	1	0	-4	-5	-4	0
(3,1,2,3,3)	0.000	1	0	-5	-4	-4	-4	0	-5	-4	0
(2,3,3,3,1)	0.000	-4	-4	-4	0	-5	1	1	1	0	0
(2,3,3,1,3)	0.000	-5	1	1	0	1	-5	-4	-4	0	0
(2,3,1,3,3)	0.000	1	-5	-4	0	-4	-4	-5	-4	0	0
(2,1,3,3,3)	0.000	-4	-4	-5	0	-4	-4	-4	-5	0	0
(1,3,3,3,2)	0.000	-4	-4	-4	-5	0	1	1	1	0	0
(1,3,3,2,3)	0.000	0	1	1	1	0	1	0	-4	-4	0
(1,3,2,3,3)	0.000	1	0	-4	-5	-4	-4	0	-4	-5	0
(1,2,3,3,3)	0.000	-4	-4	0	-5	-4	-4	-4	0	-5	0
(3,3,2,2,2)	0.000	0	0	-4	-4	0	0	0	-4	-4	-5
(3,2,3,2,2)	0.000	-5	-4	0	-4	0	0	-4	0	-4	-5
(3,2,2,3,2)	0.000	-4	-5	-5	-4	-5	-4	0	0	-4	-5
(3,2,2,2,3)	0.000	0	0	0	-4	-4	-5	-5	-5	-4	-5
(2,3,3,2,2)	0.000	-5	-4	-4	0	0	0	-4	-4	0	-5
(2,3,2,3,2)	0.000	-4	-5	-4	-5	-5	-4	0	-4	0	-5
(2,3,2,2,3)	0.000	0	0	-4	0	-4	-5	-5	-4	-5	-5
(2,2,3,3,2)	0.000	-4	-4	-5	-5	-5	-4	-4	0	0	-5
(2,2,3,2,3)	0.000	-5	-4	0	0	-4	-5	-4	-5	-5	-5
(2,2,2,3,3)	0.000	-4	-5	-5	-5	-4	-4	-5	-5	-5	-5

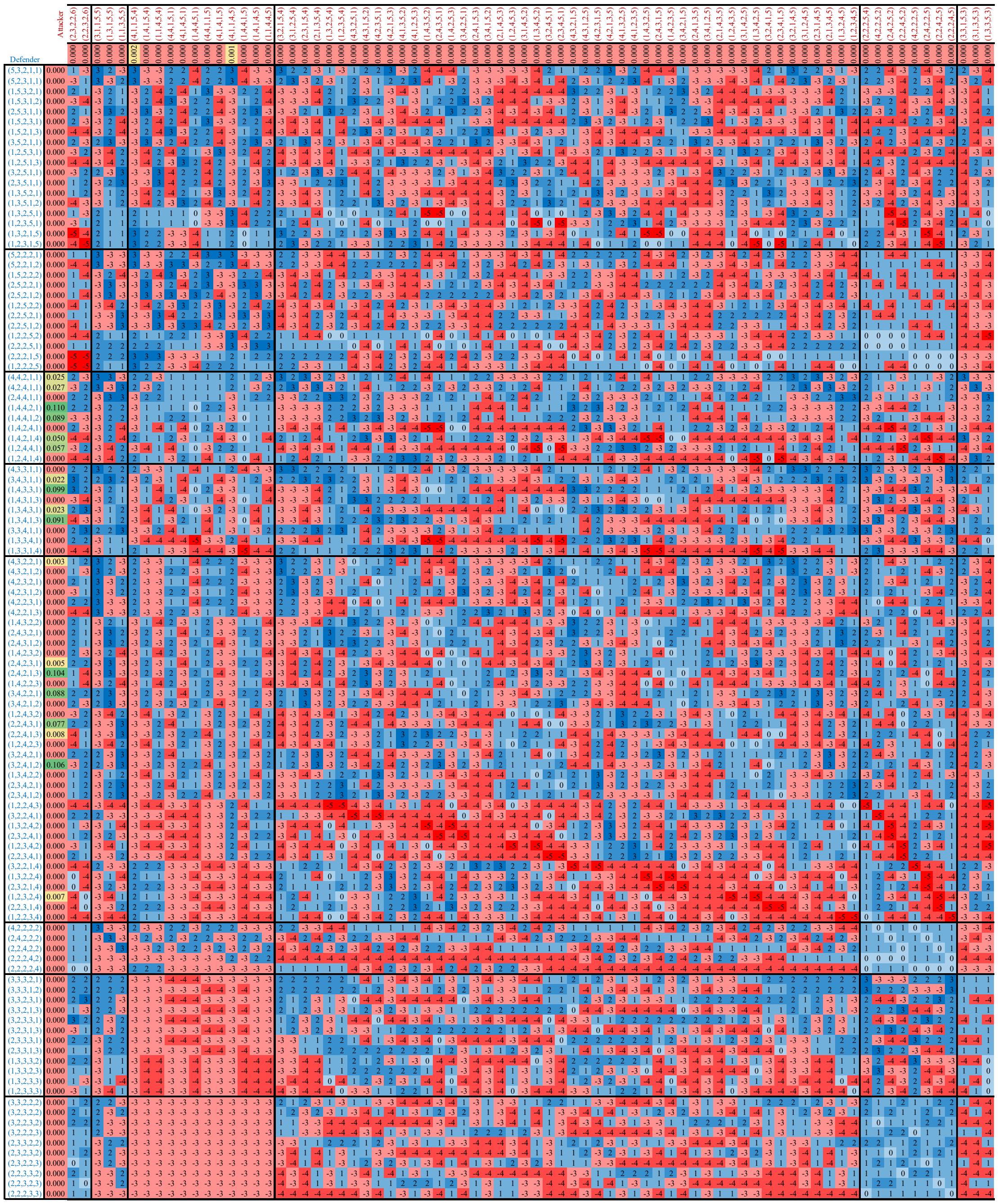
Section 16 of 16

Reduced model.

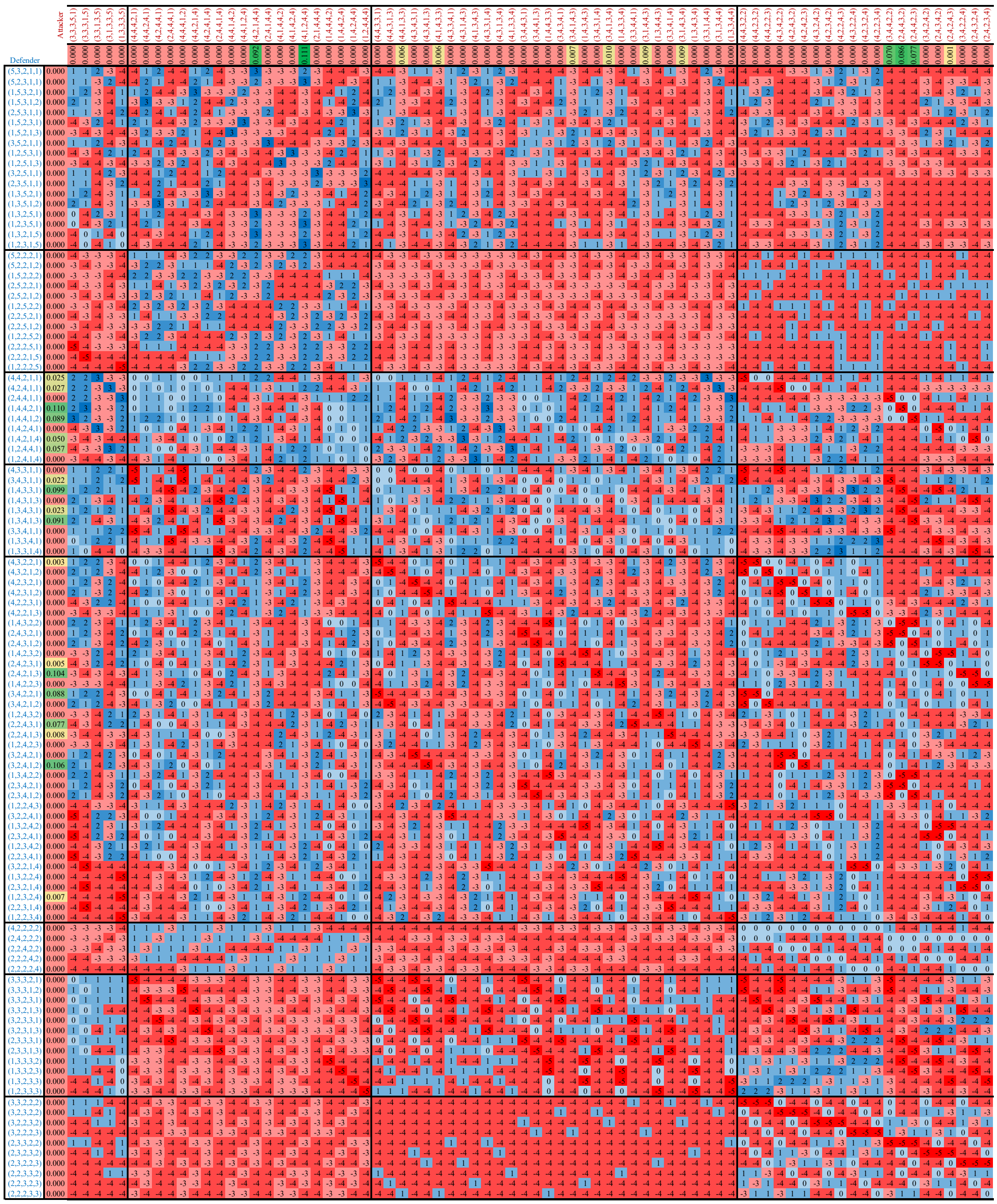
A detailed version of the reduced model, depicted in low-resolution in Figure 20 on page 70, follows. These four sections of the reduced payoff matrix comprise the full matrix depicted in Figure 20. Although difficult to read in printed form, the electronic version of this thesis contains all of the matrix data and can be much more easily read. This reduced model is based on the feature space reduction methods employed in the analysis and intelligence updates described in the scenario. This output is a payoff matrix of size $j = 111$ pure strategies and $k = 258$ pure strategies. They are labeled by pure strategy and include the mixed strategies for the game.



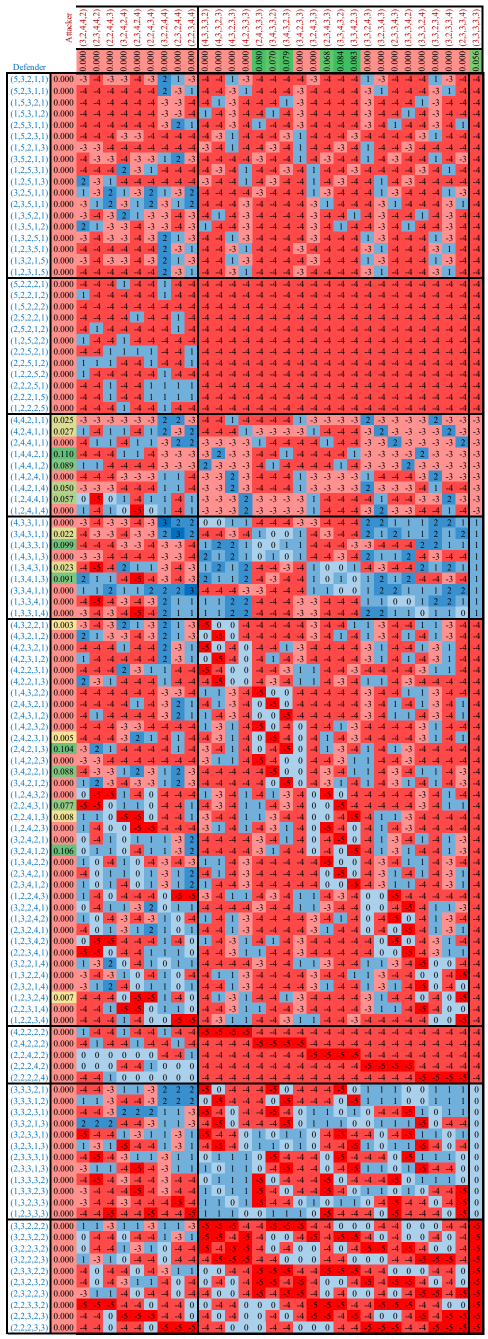
Section 1 of 4



Section 2 of 4



Section 3 of 4



Section 4 of 4

Appendix B. Excel Code

The following code is used in various Excel cells to create the Blotto payoff matrices that are analyzed by planners to understand the game. Just as in Excel, colored text is used to denote the referenced cell column letters and row numbers:

- `Front`n references the column letter of the cell containing the number of resources Player 1 assigned to Front n.
- `Player1` references the row number of the cell containing the number of resources Player 1 assigned to Front n.
- `Player2` references the column letter of the cell containing the number of resources Player 2 assigned to Front n.
- `Front`n references the row number of the cell containing the number of resources Player 2 assigned to Front n.
- `Player1Weight` references the column letter of the cell containing the weight Player 1 assigned to Front n.
- `Front`n`Weight` references the row number of the cell containing the weight Player 1 assigned to Front n.
- `Player2Weight` references the column letter of the cell containing the weight Player 1 assigned to Front n.
- `Front`n`Weight` references the row number of the cell containing the weight Player 1 assigned to Front n.

Two Fronts.

```
=IF(
(IF($Front1Player1>=Player2$Front1,1,-1)
+IF($Front2Player1>=Player2$Front2,1,-1))
>=0,
(IF($Front1Player1>Player2$Front1,$Player1Weight$Front1Weight,0)
+IF($Front2Player1>Player2$Front2,$Player1Weight$Front2Weight,0)
),
(IF($Front1Player1>Player2$Front1,0,-$Player2Weight$Front1Weight)
+IF($Front2Player1>Player2$Front2,0,-$Player2Weight$Front2Weight)
))
```

Three Fronts.

```
=IF(
(IF($Front1Player1>=Player2$Front1,1,-1)
+IF($Front2Player1>=Player2$Front2,1,-1)
+IF($Front3Player1>=Player2$Front3,1,-1))
>=0,
(IF($Front1Player1>Player2$Front1,$Player1Weight$Front1Weight,0)
+IF($Front2Player1>Player2$Front2,$Player1Weight$Front2Weight,0)
+IF($Front3Player1>Player2$Front3,$Player1Weight$Front3Weight,0)
),
(IF($Front1Player1>Player2$Front1,0,-$Player2Weight$Front1Weight)
+IF($Front2Player1>Player2$Front2,0,-$Player2Weight$Front2Weight)
+IF($Front3Player1>Player2$Front3,0,-$Player2Weight$Front3Weight)
))
```

Four Fronts.

```
=IF(
(IF($Front1Player1>=Player2$Front1,1,-1)
+IF($Front2Player1>=Player2$Front2,1,-1)
+IF($Front3Player1>=Player2$Front3,1,-1)
+IF($Front4Player1>=Player2$Front4,1,-1)
)>=0,
(IF($Front1Player1>Player2$Front1,$Player1Weight$Front1Weight,0)
+IF($Front2Player1>Player2$Front2,$Player1Weight$Front2Weight,0)
+IF($Front3Player1>Player2$Front3,$Player1Weight$Front3Weight,0)
+IF($Front4Player1>Player2$Front4,$Player1Weight$Front4Weight,0)
),
(IF($Front1Player1>Player2$Front1,0,-$Player2Weight$Front1Weight)
+IF($Front2Player1>Player2$Front2,0,-$Player2Weight$Front2Weight)
+IF($Front3Player1>Player2$Front3,0,-$Player2Weight$Front3Weight)
+IF($Front4Player1>Player2$Front4,0,-$Player2Weight$Front4Weight)
))
```

Five Fronts.

```
=IF(
(IF($Front1Player1>=Player2$Front1,1,-1)
+IF($Front2Player1>=Player2$Front2,1,-1)
+IF($Front3Player1>=Player2$Front3,1,-1)
+IF($Front4Player1>=Player2$Front4,1,-1)
+IF($Front5Player1>=Player2$Front5,1,-1)
)>=0,
(IF($Front1Player1>Player2$Front1,$Player1Weight$Front1Weight,0)
+IF($Front2Player1>Player2$Front2,$Player1Weight$Front2Weight,0)
+IF($Front3Player1>Player2$Front3,$Player1Weight$Front3Weight,0)
+IF($Front4Player1>Player2$Front4,$Player1Weight$Front4Weight,0)
+IF($Front5Player1>Player2$Front5,$Player1Weight$Front5Weight,0)
),
(IF($Front1Player1>Player2$Front1,0,-$Player2Weight$Front1Weight)
+IF($Front2Player1>Player2$Front2,0,-$Player2Weight$Front2Weight)
+IF($Front3Player1>Player2$Front3,0,-$Player2Weight$Front3Weight)
+IF($Front4Player1>Player2$Front4,0,-$Player2Weight$Front4Weight)
+IF($Front5Player1>Player2$Front5,0,-$Player2Weight$Front5Weight)
))
```

Bibliography

1. Ahmadinejad, AmirMahdi, Sina Dehghani, MohammadTaghi Hajiaghayi, Brendan Lucier, Hamid Mahini, and Saeed Seddighin. "From Duels to Battefields: Computing Equilibria of Blotto and Other Games". *Thirtieth AAAI Conference on Artificial Intelligence*. 2016.

2. Armistead, Edwin Leigh (editor). *Information Operations: Warfare and the Hard Reality of Soft Power*. Potomac Books, Washington DC, 1st edition, 2004.

3. Arreguín-Toft, Ivan. *How the Weak Win Wars: A Theory of Asymmetric Conflict*. Cambridge University Press, Cambridge UK, 2005.

4. Aumann, B Y Robert and Adam Brandenburger. "Epistemic Conditions for Nash Equilibrium". *Econometrica*, 63(5):1161–1180, 1995.

5. Borel, Émile. "The Theory of Play and Integral Equations with Skew Symmetric Kernels". *Econometrica*, 21(1):97, 1953.

6. Box, G. E. P. "Robustness in the strategy of scientific model building". R. L. Launer and G. N. Wilkinson (editors), *Robustness in Statistics*, 201–236. Academic Press, 1979.

7. Bullock, Richard K., Richard F. Deckro, and Jeffery D. Weir. "Methodology for competitive strategy development". *Computers and Operations Research*, 35(6):1865–1873, 2008.

8. Chowdhury, Subhasish M., Dan Kovenock, and Roman M. Sheremeta. "An experimental investigation of Colonel Blotto games". *Economic Theory*, 52(3):833–861, 2013.

9. von Clausewitz, Carl. *On War*. Charles River Editors, Apple Books edition, 2011.

10. Collins, Andrew J. and Patrick T. Hester. "Colonel Blotto Games and Lancaster's Equations: A Novel Military Modeling Combination". *MODSIM World Paper 2011*, 93–99, 2011.

11. Deptula, David A. "Interdependent Warfare: Combined Effects Power in the 21st Century". *Mitchell Institute for Aerospace Studies*, 10(March):19, 2018.

12. Geoffrion, Arthur M. "The Purpose of Mathematical Programming is insight, not numbers". *Interfaces*, 7(1):81–92, 1976.

13. Gerasimov, Valery. "The Value of Science Is in the Foresight New Challenges Demand Rethinking the Forms and Methods of Carrying out Combat Operations". *Military Review*, (February):23–29, 2016.

14. Golman, Russell and Scott E. Page. "General Blotto: games of allocative strategic mismatch". *Public Choice*, 138(3-4):279–299, 2009.

15. Gross, Oliver. *An Infinite-Dimensional Extension of a Symmetric Blotto Game*. Technical report, The Rand Corporation, Santa Monica CA, 1951.

16. Gross, Oliver and Robert Wagner. *A Continuous Colonel Blotto Game*. Technical report, The Rand Corporation, Santa Monica CA, 1950.

17. Grotte, J. H. and P. S. Brooks. "Measuring Naval Presence Using Blotto Games". *International Journal of Game Theory*, 12(4):225–236, 1983.

18. Harold, Scott W, Yoshiaki Nakagawa, Junichi Fukuda, John A Davis, Keiko Kono, Dean Cheng, and Kazuto Suzuki. *The U.S.-Japan Alliance and Deterring Gray Zone Coercion in the Maritime, Cyber, and Space Domains*. Technical report, The Rand Corporation, Santa Monica CA, 2017.

19. Hart, Sergiu. "Discrete Colonel Blotto and General Lotto games". *International Journal of Game Theory*, 36(3-4):441–460, 2008.

20. Haywood, O. G. "Military Decision and Game Theory". *Journal of the Operations Research Society of America*, 2(4):365–385, 1954.

21. Hora, Stephen C. "Eliciting Probabilities from Experts". Ward Edwards, Ralph F. Jr. Miles, and Detlof von Winterfeldt (editors), *Advances in Decision Analysis: From Foundations to Applications*, chapter 8, 129–153. Cambridge University Press, Cambridge UK, 2007.

22. Joint Chiefs of Staff. *Joint Publication 5-0: Joint Planning*. Washington DC, 16 June 2017.

23. Joint Chiefs of Staff. *Joint Concept for integrated campaign planning*. Washington DC, 16 March 2018.

24. Kass, Sam. "Rock Paper Scissors Spock Lizard", 2012. URL `http://www.samkass.com/theories/RPSSL.html`.

25. Kovenock, Dan, Michael J. Mauboussin, and Brian Roberson. "Asymmetric Conflicts With Endogenous Dimensionality". *The Korean Economic Review*, 26(2):287–288, 2010.

26. Liang, Qiao and Wang Xiangsui. *Unrestricted Warfare*. PLA Literature and Arts Publishing House, Beijing CN, 1999.

27. de Marchi, Scott. *Computational and Mathematical Modeling in the Social Sciences*. Cambridge University Press, Cambridge UK, 2005.

28. McCloskey, Joseph F. "The Beginnings of Operations Research: 1934 - 1941". *Operations Research*, 35:143–152, 1987.

29. Meilinger, Phillip S. “Time in War”. *Joint Force Quarterly*, 87(4):93–100, 2017.

30. Modzelewski, Kevin, Joel Stein, and Julia Yu. *An Experimental Study of Classic Colonel Blotto Games*. Technical report, MIT, Cambridge MA, 2009.

31. Morrow, James D. *Game Theory for Political Scientists*. Princeton University Press, Princeton NJ, 1994.

32. Nash, John. “Non-Cooperative Games”. *The Annals of Mathematics*, 54(2):286, 1951.

33. von Neumann, John and Oskar Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton NJ, 3rd edition, 1944.

34. Novikov, D A. “Hierarchical Models of Warfare”. *Automation and Remote Control*, 74(10):1733–1752, 2013.

35. Prisner, Erich. *Game Theory Through Examples*. 1. The Mathematical Association of America, Washington DC, 2014.

36. Rittel, W. and M. Webber. “Dilemmas in a General Theory of Planning”. *Policy Sciences*, 4(2):155–169, 1973.

37. Roberson, Brian. “The Colonel Blotto game”. *Economic Theory*, 29(1):1–24, 2006.

38. Shubik, Martin and Robert James Weber. “Systems Defense Games: Colonel Blotto, Command and Control”. *Naval Research Logistics Quarterly*, 28(2):281–287, 1981.

39. Tangredi, Sam J. *Anti-Access Warfare: Countering A2/AD Strategies*. Naval Institute Press, Annapolis MD, 2013.

40. Tangredi, Sam J. “Antiaccess Warfare as Strategy”. *Naval War College Review*, 71(1):33–51, 2018.

41. Turban, Efraim, Jay E. Aronson, and Ting-Peng Liang. *Decision Support Systems and Intelligent Systems*. Pearson, Upper Saddle River NJ, 6th edition, 2001.

42. U.S. Army Training and Doctrine Command. *TRADOC Pamphlet 525-3-1: The U.S. Army in Multi-Domain Operations 2028*. Fort Eustis VA, 6 December 2018.

43. Watson, Joel. *Strategy*. W. W. Norton & Company, Inc., New York NY, 2013.

Vita

Captain John Andrew Schlicht enlisted in the United States Army in 2007 and earned a commission in the United States Army Corps of Engineers in May 2008 from the Army Infantry School's Officer Candidate School. He holds a Bachelor of Science Degree in Management from Purdue University and a Master of Science Degree in Engineering Management from Missouri University of Science and Technology.

Captain Schlicht commanded the Headquarters Company, 130th Theater Engineer Brigade (TEB), from January 2015 to April 2016. Following command, he was accepted into the Army's Functional Area 49, Operations Research / Systems Analysis. He will graduate with a Master of Science Degree in Operations Research from the Air Force Institute of Technology in March 2019.

Captain Schlicht has deployed twice to Afghanistan, serving as the Company Intelligence Officer and a Sapper Platoon Leader in 2010-11, and then as a TEB Planner, the Theater's Engineer Force Manager, and the TEB Chief of Operations in 2013-14. He has also deployed to Central America as the Joint Task Force - Bravo Army Forces Battalion Chief of Operations in 2012.

Captain Schlicht's military awards and decorations include the Bronze Star Medal, Meritorious Service Medal (1 OLC), Joint Service Commendation Medal, Army Commendation Medal (1 OLC), National Defense Service Medal, Afghanistan Campaign Medal (2 Service Stars), Global War on Terrorism Service Medal, Military Outstanding Volunteer Service Medal, Army Service Ribbon, Army Overseas Service Ribbon (Numeral 3 Device), NATO Medal (ISAF), Combat Action Badge, Parachutist Badge, Presidential Unit Citation, Valorous Unit Award, Meritorious Unit Commendation, Army Superior Unit Award (1 OLC), and the German Armed Forces Badge for Military Proficiency in Gold.

REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704–0188

1. REPORT DATE *(DD–MM–YYYY)*	2. REPORT TYPE	3. DATES COVERED *(From — To)*
21–03–2019	Master's Thesis	Sept 2017 — Mar 2019

4. TITLE AND SUBTITLE

OPERATIONS RESEARCH METHODS FOR MULTI-DOMAIN CAMPAIGN PHASE PLANNING

5a. CONTRACT NUMBER

5b. GRANT NUMBER

5c. PROGRAM ELEMENT NUMBER

6. AUTHOR(S)

John Andrew Schlicht

5d. PROJECT NUMBER

5e. TASK NUMBER

5f. WORK UNIT NUMBER

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)

Air Force Institute of Technology
Graduate School of Engineering and Management (AFIT/EN)
2950 Hobson Way
WPAFB OH 45433-7765

8. PERFORMING ORGANIZATION REPORT NUMBER

AFIT-ENS-MS-19-M-148

9. SPONSORING / MONITORING AGENCY NAME(S) AND ADDRESS(ES)

Joint Warfare Analysis Center
Attn: Ms. Jill S. Morrissett
4048 Higley Road
Dahlgren, VA 22448
DSN 249-5754, COMM 540-653-5754
Email: jill.s.morrissett.civ@mail.mil

10. SPONSOR/MONITOR'S ACRONYM(S)

JWAC

11. SPONSOR/MONITOR'S REPORT NUMBER(S)

12. DISTRIBUTION / AVAILABILITY STATEMENT

DISTRIBUTION STATEMENT A:
APPROVED FOR PUBLIC RELEASE; DISTRIBUTION UNLIMITED.

13. SUPPLEMENTARY NOTES

14. ABSTRACT

In "Antiaccess Warfare as Strategy," Tangredi posits the question and need to consider multiple domains and governmental and warfighting functions in various phases of campaign execution. Multi-domain integration within and across various phases of the joint campaign presents a host of non-linear factors that are compounded and amplified by uncertainties. Colonel Blotto is a simple game that is suited to compare traditional force-on-force military engagements where mass wins the day, but have had limited application to more complex military planning. This thesis explores the formulation schema, data-driven parameters, methods of calculation, and scenarios applicable to a generalized Colonel Blotto (General Blotto) game. It explores this generalized game theory framework, its applicability to multi-domain operations, and recommends future research areas that could help to extend its applicability, enabling planners and commanders to gain similar insight as to those straight-forward applications.

15. SUBJECT TERMS

Colonel Blotto, General Blotto, Game Theory, Campaign Planning, Antiaccess, Multi-Domain

16. SECURITY CLASSIFICATION OF: a. REPORT	b. ABSTRACT	c. THIS PAGE	17. LIMITATION OF ABSTRACT	18. NUMBER OF PAGES	19a. NAME OF RESPONSIBLE PERSON
U	U	U	U	126	Dr. Richard F. Deckro, AFIT/ENS
					19b. TELEPHONE NUMBER *(include area code)* 937-255-6565 x4325; richard.deckro@afit.edu